

Symmetries and Conserved Quantities

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Review

- In the previous lecture, we discussed the action for the complex scalar field; in Minkowski space, we have

$$S = - \int_x \left[\partial_\mu \phi \partial^\mu \phi^* + m^2 \phi \phi^* + 4\lambda (\phi \phi^*)^2 \right]. \quad (8.1)$$

- We noted that the action has an additional symmetry under the transformation

$$\phi(x) \rightarrow e^{i\alpha} \phi(x), \quad (\phi^*(x) \rightarrow e^{-i\alpha} \phi^*(x)), \quad (8.2)$$

with arbitrary (but constant) α

- Let us now discuss the consequence of this symmetry

Classical Equations of Motion

- Let's write $S = \int_x \mathcal{L}$ with $\mathcal{L}(\phi, \phi^*)$ the Lagrangian density
- We can get the *classical* equations of motion from requiring S to be invariant under changes of ϕ : $S \rightarrow S' = S$ or $\delta S = 0$
- Treating ϕ, ϕ^* as independent fields,

$$\delta S = \int_x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \phi^*} \delta \phi^* + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \delta \partial_\mu \phi^* \right]. \quad (8.3)$$

- Using the chain rule, we can write

$$\int_x \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi = \int_x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right] - \int_x \delta \phi \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right], \quad (8.4)$$

and similarly for ϕ^*

Classical Equations of Motion

- The *classical* equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 = -m^2 \phi^* - 8\lambda(\phi\phi^*)\phi^* + \square\phi^*, \quad (8.5)$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} = 0 = -m^2 \phi - 8\lambda(\phi\phi^*)\phi + \square\phi. \quad (8.6)$$

- If we *use* these classical equations of motion, we get

$$\delta S = \int_x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \delta\phi^* \right]. \quad (8.7)$$

- As long as the integrand is well-behaved at infinity, the integral of a total derivative vanishes
- We therefore have

$$\delta S = 0 \quad (8.8)$$

for *any* small variation $\delta\phi$ that's well-behaved at infinity

Symmetry and Conserved Quantities

- Now consider putting the system into a box; the equations of motion (8.5) still hold, and we find

$$\delta S_{\text{box}} = \int_{\text{box}} d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \delta \phi^* \right]. \quad (8.9)$$

- For the symmetry (8.2), the action is invariant, hence $\delta S_{\text{box}} = 0$
- Plugging in $\delta \phi = i\alpha \phi(x)$ from (8.2), this gives

$$0 = \alpha \int_{\text{box}} d^4x \partial_\mu \left[i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \phi - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \phi^* \right]. \quad (8.10)$$

- Since the shape and size of the box are arbitrary, the integrand must vanish:

$$\partial_\mu \left[i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \phi - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \phi^* \right] = 0 \quad (8.11)$$

Symmetry and Conserved Quantities

- Let's denote the term in brackets in (8.11) as

$$j^\mu(x) = i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \phi - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \phi^* \quad (8.12)$$

- Using the explicit action (8.1), j^μ becomes

$$j^\mu = i\phi^* \partial^\mu \phi - \text{c.c.} = 2\text{Im}\phi \partial^\mu \phi^*, \quad (8.13)$$

where c.c. denotes the complex conjugate

- With this notation, (8.11) becomes the classical conservation law for the current density j^μ :

$$\partial_\mu j^\mu = 0 \quad (8.14)$$

Symmetry and Conserved Quantities

- Writing $j^\mu = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix}$ we have

$$\partial_\mu j^\mu = \partial_0 \rho + \nabla \cdot \mathbf{j} = 0. \quad (8.15)$$

- Integrating over an infinite spatial volume $\int d^3x$ we have

$$\partial_0 \int d^3x \rho = 0, \quad (8.16)$$

where we have assumed that \mathbf{j} falls sufficiently fast at infinity

Symmetry and Conserved Quantities

- We found that the symmetry (8.2) gives rise to a *conserved charge*

$$Q \equiv \int d^3x \rho \quad (8.17)$$

- This conservation law is *classical*
- The associated current density j^μ is called the *Noether current*
- The *Noether theorem* stating that **every continuous symmetry corresponds to a conserved quantity** is an extremely important result in physics
- While we only discussed the particular case (8.2), another important case is the invariance of S under translations, giving rise to the **conservation of energy** and **conservation of momentum**