Towards General Relativity

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Lessons from Special Relativity

- Time and space belong together – "spacetime"; 4-vectors x^μ, etc
- Physics does not depend on the coordinates chosen
- In particular, certain observables are invariant under **linear** coordinate transformations $dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$: e.g.

$$ds'^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{9.1}$$

- Invariance gise rise to conserved quantities; e.g.

$$\partial_\mu T^{\mu\nu} = 0\,, \quad \partial_\mu J^\mu = 0\,. \tag{9.2}$$

Towards General Relativity

- Coordinate Transformations:

$$x^\mu \to x'^\mu = x'^\mu \left(x^\mu\right) . \tag{9.3}$$

- Special Relativity: **linear** coordinate transformations $dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$
- We can **generalize** this to **non-linear** coordinate transformations $x' = x'(x)$
- Consequence: the transformation matrix

$$R^\mu_\nu = \frac{\partial x^\mu}{\partial x'^\mu} , \tag{9.4}$$

can now be a **function of coordinates** (while $\Lambda^\mu{}_\nu$ just numbers)

Example: Polar Coordinates

- Let's do a simple example of a non-linear coordinate transformation
- To minimize the writing, let's pick 2-dimensional Euclidean space with coordinates $x^i = \begin{pmatrix} x \\ y \end{pmatrix}$
- We now want to transform from Euclidean coordinates x^i to polar coordinates r, ϕ
- We write

$$x'^i = \begin{pmatrix} r \\ \phi \end{pmatrix}, \quad x^i = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}. \tag{9.5}$$

- Because the dependence of x on x' (and vice versa) is not linear, this is an example of a non-linear coordinate transformation

Example: Polar Coordinates

- We write

$$x'^i = \begin{pmatrix} r \\ \phi \end{pmatrix}, \quad x^i = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}. \tag{9.6}$$

- Let us calculate the transformation matrix for this coordinate transformation
- We have

$$R^i{}_j \equiv \frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}. \tag{9.7}$$

- Note that $R^i{}_j$ depends on r, ϕ

Example: Polar Coordinates

- Seems like non-linear coordinate transformations are much more complicated than linear coordinate-transformations
- This is true; as a consequence GR is much more complicated than SR
- Nevertheless, some quantities are **invariant** under non-linear coordinate transformations, just like what was the case in SR
- For our example, take the (infinitesimal) length of a 2-vector:

$$|d\vec{x}|^2 = dx^i dx^j \delta_{ij} \to R^i{}_m R^j{}_n \delta_{ij} dx'^m dx'^n = dx'^m dx'^n \delta_{mn}\,, \qquad (9.8)$$

because the Euclidean metric δ_{ij} is invariant under rotations

General Coordinate Transformations

- Special Relativity is built on invariance wrt **linear transformations**

$$dx^\mu = \Lambda^\mu{}_\nu dx'^\nu \,. \tag{9.9}$$

- **General Relativity** is built on invariance wrt **non-linear transformations**

$$dx^\mu \equiv \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu = R^\mu{}_\nu dx'^\nu \,. \tag{9.10}$$

- Equivalently:

$$dx'^\mu \equiv \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = \frac{1}{\frac{\partial x^\nu}{\partial x'^\mu}} dx^\nu = \left(R^{-1}\right)^\mu{}_\nu \, dx'^\nu \,. \tag{9.11}$$

Example: Polar Coordinates

- Back to our polar coordinate example
- From (9.8), (9.7) we have

$$g'_{mn} = R^i_m R^j_n \delta_{ij} = \mathbf{R}^T \cdot \mathbf{1} \cdot \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} . \qquad (9.12)$$

- Note: **New metric depends on coordinates!**
- Note: Transformed metric is **not** Kronecker symbol!
- Length of a vector in new coordinate system:

$$d\vec{x}'^2 = dx'^i dx'^j g'_{ij} = dr^2 + r^2 d\phi^2 \qquad (9.13)$$

- Compare to original coordinate system:

$$d\vec{x}^2 = dx^2 + dy^2 = (d(r\cos\phi))^2 + (d(r\sin\phi))^2 = dr^2 + r^2 d\phi^2 \qquad (9.14)$$

- Length of vector is invariant!

Scalars, Vectors and Tensors

- We can define scalars, vectors and tensors under **general coordinate transformations** $x \to x'(x)$:

$$S \to S' = S\,, \quad V^\mu \to V'^\mu = R^\mu_{\ \nu} V^\nu\,, \quad X^{\mu\nu} \to X'^{\mu\nu} = R^\mu_{\ \alpha} R^\nu_{\ \beta} X^{\alpha\beta}\,. \tag{9.15}$$

- A different name for a **general coordinate transformation** is **diffeomorphism**
- Generalizes the transformations of special relativity
- Same guiding principle: physics is independent from coordinates used
- Lorentz Covariance $\longrightarrow$ General Covariance