The Geodesic Equation

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Spring 2021

From Coordinate Transformations to Gravity

- Last lecture, we introduced **general covariance**, the notion that physics does not depend on (arbitrarily chosen) coordinates
- By itself, this is just an exercise in math
- In this lecture, we'll take the first step towards considering **physical consequences** of taking the math seriously
- The procedure is akin to our treatment of **special relativity**
- In SR, we considered linear coordinate transformations (math); this led to physics consequences (e.g. time-delay)

Particle in Minkowski spacetime

Consider a massive particle in Minkowski spacetime:

- No forces
- Particle moves on a "straight line"
- Proper time for particle:

$$d\tau'^2 = -ds^2 = -g'_{\mu\nu} dx'^\mu dx'^\nu \tag{10.1}$$

- Absence of force means particle moves with constant velocity:

$$\frac{dx'^\mu}{d\tau'} = \mathrm{const}\,. \tag{10.2}$$

- Take another derivative wrt to proper time:

$$\frac{d^2 x'^\mu}{d\tau'^2} = 0\,. \tag{10.3}$$

Einstein and the Elevator

Einstein:

- Enclosed in an elevator, you can't tell if it's accelerating or if some alien has messed with gravity
- Locally, gravity is **indistinguishable** from acceleration
- But there is a non-linear coordinate transformation from Minkowski space x' to accelerated coordinate system x:

$$x'^{\mu} = x'(x)\,, \quad \text{where } x'^{\mu} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}\,. \tag{10.4}$$

- In Minkowski space

$$\frac{d^2 x'^{\mu}}{d\tau'^2} = 0\,, \tag{10.5}$$

  so this equation must hold true
- Let's transform (10.5) to coordinates x^{μ}:

General Coordinate Transformations

- We perform (non-linear) transform from Minkowski coordinate x'^{μ} to new coordinates x^{μ}
- The transformation matrix is

$$R^{\mu}_{\ \nu} \equiv \frac{\partial x'^{\mu}}{\partial x^{\nu}}\,. \tag{10.6}$$

- The new metric $g_{\mu\nu}$ will (in general) be non-Minkowksi
- But we can use that scalars are invariant under general coordinate transformations, e.g.

$$d\tau'^2 = -ds'^2 = -ds^2 = d\tau^2\,, \tag{10.7}$$

hence $d\tau' = d\tau$.

General Coordinate Transformations

We thus have

$$
\begin{aligned}
0 &= \frac{d^2 x'^\mu}{d\tau'^2} = \frac{d^2 x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right), \\
&= \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\nu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\right),
\end{aligned}
$$

General Coordinate Transformations

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\begin{aligned}
0 &= \frac{d^2 x'^\mu}{d\tau'^2} = \frac{d^2 x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right), \\
&= \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\alpha}{d\tau}\frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\alpha}\right), \\
&= \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}, \\
&= \frac{\partial x'^\mu}{\partial x^\nu}\left[\frac{d^2 x^\nu}{d\tau^2} + \frac{\partial x^\nu}{\partial x'^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}\right], \\
&= R^\mu_{\ \nu}\left[\frac{d^2 x^\nu}{d\tau^2} + \frac{\partial x^\nu}{\partial x'^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}\right], \qquad (10.8)
\end{aligned}
$$

The Geodesic Equation

- The transformation matrix is non-singular, so (10.5) in coordinates x^μ becomes
$$\frac{d^2x^\nu}{d\tau^2} + \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{10.9}$$

- Looks complicated, let's introduce the new notation
$$\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} \equiv \Gamma^\nu_{\alpha\beta}\,. \tag{10.10}$$

- With these new symbols, we have
$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{10.11}$$

- This is the **geodesic equation**

Vectors and such

- Recall my warning about things that look like but are not vectors in lecture 4
- Take $\frac{d^2x'^\mu}{d\tau'^2}$ as an example
- It carries one index, so one might be tempted to assume it is a vector under general coordinate transformations
- If it **was a vector**, then it would have to transform as

$$\frac{d^2x'^\mu}{d\tau'^2} \to R^\mu_{\ \nu} \frac{d^2x^\mu}{d\tau^2} \tag{10.12}$$

- However, we found by direct calculation in (10.11) that there is an additional piece proportional to $\Gamma^\nu_{\alpha\beta}$
- **This implies that $\frac{d^2x^\mu}{d\tau^2}$ is not a vector under diffeomorphisms!**