Meaning of The Geodesic Equation

paul.romatschke@colorado.edu

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Review

- In the last lecture, we derived the **geodesic equation**

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{11.1}$$

- In this lecture we want to answer 2 questions:
  - What does the geodesic equation mean?
  - How does one calculate the mysterious symbols $\Gamma^\nu_{\alpha\beta}$?
- Let's get started!

Proper time

- Proper time increment for a particle is

$$d\tau = \sqrt{-ds^2} = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}\,. \tag{11.2}$$

- Get proper time duration by integration:

$$\tau = \int_A^B d\tau = \int_A^B \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}\,. \tag{11.3}$$

- Can pull out "line element" ds:

$$\tau = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}ds = \int_{s_A}^{s_B} ds\sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}\,. \tag{11.4}$$

Equations of motion

- Recall that for Lagrangian mechanics, we got EoM for minimal action
- Let us now calculate the EoM that result from **minimizing proper time** τ for a particle traveling from A to B
- Let's treat τ to be like an action and consider coordinate variation

$$x^\mu \to x^\mu + \delta x^\mu \,, \quad g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \,, \quad \delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \delta x^\sigma \qquad (11.5)$$

- Setting the variation of τ to zero we get

$$0 = \delta\tau = \int_{s_A}^{s_B} ds \frac{1}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}} \left[-\delta\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)\right] . \qquad (11.6)$$

Equations of motion

- To simplify the expression, replace again $\sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} = \frac{d\tau}{ds}$ to find

$$0 = \delta\tau = \frac{1}{2}\int_{\tau_A}^{\tau_B} d\tau \frac{ds^2}{d\tau^2}\left[-\delta\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)\right]. \tag{11.7}$$

- For the variation we get

$$\delta\left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = \delta g_{\mu\nu}\left(\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) + g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\nu}{ds}\frac{dx^\mu}{ds},$$

$$= \partial_\sigma g_{\mu\nu}\delta x^\sigma \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\nu}{ds}\frac{dx^\mu}{ds}. \tag{11.8}$$

Equations of motion

- Using the measure $\frac{ds^2}{d\tau^2}$ to replace all $\frac{d}{ds}$ by $\frac{d}{d\tau}$ we get

$$0 = \int_{\tau_A}^{\tau_B} d\tau \left[\partial_\sigma g_{\mu\nu}\delta x^\sigma \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d\delta x^\nu}{d\tau}\frac{dx^\mu}{d\tau}\right]. \tag{11.9}$$

- Integrating by parts for the terms containing $\frac{d}{d\tau}\delta x^{\mu,\nu}$ we have (neglecting boundary terms)

$$\begin{aligned} 0 &= \int_{\tau_A}^{\tau_B} d\tau \left[\partial_\sigma g_{\mu\nu}\delta x^\sigma \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \delta x^\mu \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\right. \\ &\quad \left. -\delta x^\nu \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\right)\right]. \end{aligned} \tag{11.10}$$

Equations of motion

- Using the chain rule we have

$$0 = \int_{\tau_A}^{\tau_B} d\tau \left[\partial_\sigma g_{\mu\nu}\delta x^\sigma \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} - \delta x^\nu \frac{dx^\mu}{d\tau}\frac{dg_{\mu\nu}}{d\tau} - \delta x^\mu \frac{dx^\nu}{d\tau}\frac{dg_{\mu\nu}}{d\tau}\right]. \quad (11.11)$$

- Writing $\frac{dg_{\mu\nu}}{d\tau} = \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{d\tau}$ this becomes

$$0 = \int_{\tau_A}^{\tau_B} d\tau \left[\partial_\sigma g_{\mu\nu}\delta x^\sigma \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} - \delta x^\nu \frac{dx^\mu}{d\tau}\frac{dx^\beta}{d\tau}\partial_\beta g_{\mu\nu} - \delta x^\mu \frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\partial_\alpha g_{\mu\nu}\right] \quad (11.12)$$

Equations of motion

- Relabeling indices this becomes

$$0 = \int_{\tau_A}^{\tau_B} d\tau \delta x^\mu \left[-2 g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \right. \tag{11.13}$$
$$\left. + \partial_\mu g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \partial_\beta g_{\nu\mu} \frac{dx^\nu}{d\tau} \frac{dx^\beta}{d\tau} - \partial_\alpha g_{\mu\nu} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} \right] .$$

- ...which simplifies to...

$$0 = \int_{\tau_A}^{\tau_B} d\tau \delta x^\mu \left[2 g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \left(-\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\mu\beta} \right) \right]$$

- Since δx is arbitrary, the integral vanishes only if the integrand itself is zero

Equations of motion

- We therefore get the **equations of motion** for **minimal proper time between** A, B as

$$0 = g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(-\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\mu\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\,. \quad (11.14)$$

- Contracting this equation with $g^{\sigma\mu}$ then leads to

$$0 = \frac{d^2x^\sigma}{d\tau^2} + \frac{1}{2}g^{\mu\sigma}\left(-\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\mu\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\,. \quad (11.15)$$

- This is nothing but the **geodesic equation** (11.1) with coefficients

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}\left(-\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\mu\beta}\right) \quad (11.16)$$

Meaning of the geodesic equation

- We found that the geodesic equation is the **equation of motion** for a particle to take **minimum proper time** to travel from A to B
- We found that the coefficients $\Gamma^{\sigma}_{\alpha\beta}$ **can be calculated directly from the metric tensor**
- The coefficients $\Gamma^{\sigma}_{\alpha\beta}$ are called **connection coefficients** or **Christoffel symbols**
- We will calculate $\Gamma^{\sigma}_{\alpha\beta}$ for some examples in the next lecture