Christoffel Symbols

paul.romatschke@colorado.edu

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Review

- In the last lectures, we discussed the **geodesic equation**

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad (12.1)$$

- We derived an expression for the Christoffel symbols $\Gamma^\nu_{\alpha\beta}$ in terms of the metric tensor in the last lecture:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}\left(-\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta}\right) \qquad (12.2)$$

- In this lecture, we calculate examples for Christoffel symbols

Christoffels

- Have another look at the definition of the Christoffel symbols:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(-\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta}\right) \quad (12.3)$$

- You should note that these are **symmetric** in the indices α, β
- In total, the Christoffel's have three indices, so in 4D Minkowski spacetime, they have $4 \times 4 \times 4 = 64$ components
- Because of the symmetry in the lower indices, only $4 \times 10 = 40$ components are independent
- That's still many components, so calculating the Christoffels is hard work

Christoffels for Minkowski space

- The metric for Minkowski space is simply a diagonal matrix $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$
- The Minkowski metric has constant coefficients, so all derivatives wrt coordinates vanish
- Since $\Gamma^{\mu}_{\alpha\beta}$ contains only derivatives of the metric, we get

$$\Gamma^{\mu}_{\alpha\beta} = 0\,, \tag{12.4}$$

  for Minkowski space
- As a consequence, the geodesic equation (12.1) in Minkowski space becomes

$$\frac{d^2 x^{\mu}}{d\tau^2} = 0\,. \tag{12.5}$$

- This is just the statement that a particle with no forces travels along a straight line with constant velocity!

Polar Coordinates

- As our second example, let's do polar coordinates $x^\mu = \begin{pmatrix} r \\ \phi \end{pmatrix}$
- We calculated the metric for polar coordinates in lecture 9, specifically Eq. (9.12):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} . \tag{12.6}$$

(Note that in a slight abuse of notation, I'm writing Greek indices here even though the metric is positive definite)

- Let us now calculate the Christoffels for this metric

Polar Coordinates

- Because of the symmetry in lower indices, and because we are in two rather than 4 dimensions, there are only $2 \times 3 = 6$ independent components for $\Gamma^{\mu}_{\alpha\beta}$
- Explicitly, these are

$$\Gamma^{r}_{rr}\,,\Gamma^{r}_{r\phi}\,,\Gamma^{r}_{\phi\phi}\,,\Gamma^{\phi}_{rr}\,,\Gamma^{\phi}_{r\phi}\,,\Gamma^{\phi}_{\phi\phi}\,. \tag{12.7}$$

- Let's calculate them! Picking first

$$\Gamma^{r}_{rr} = \frac{1}{2} g^{r\gamma}\left(\partial_r g_{\gamma r} + \partial_r g_{\gamma r} - \partial_\gamma g_{rr}\right)\,, \tag{12.8}$$

we find that we need the inverse metric $g^{\mu\nu}$ for polar coordinates

Polar Coordinates

- Since the metric in polar coordinates is diagonal, this is easy:

$$g^{\mu\nu} = \left(g^{-1}\right)^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} . \tag{12.9}$$

- The inverse metric is diagonal, so our expression (12.8) simplifies to

$$\Gamma^r_{rr} = \frac{1}{2}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{1}{2}\partial_r g_{rr} , \tag{12.10}$$

- However, we have $g_{rr} = 1$, so the derivative vanishes, and we get

$$\Gamma^r_{rr} = 0 . \tag{12.11}$$

Polar Coordinates

- Next up is

$$\Gamma^r_{r\phi} = \frac{1}{2} g^{r\gamma} \left(\partial_r g_{\gamma\phi} + \partial_\phi g_{\gamma r} - \partial_\gamma g_{r\phi}\right) . \tag{12.12}$$

- Since the metric is diagonal, this simplifies to

$$\Gamma^r_{r\phi} = \frac{1}{2} \left(\partial_r g_{r\phi} + \partial_\phi g_{rr} - \partial_r g_{r\phi}\right) = \frac{1}{2} \partial_\phi g_{rr} . \tag{12.13}$$

- However, g_{rr} is constant, so the derivative again vanishes, leading to

$$\Gamma^r_{r\phi} = 0 . \tag{12.14}$$

Polar Coordinates

- Next up is

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{r\gamma} \left(\partial_\phi g_{\gamma\phi} + \partial_\phi g_{\gamma\phi} - \partial_\gamma g_{\phi\phi}\right) . \tag{12.15}$$

- Since the metric is diagonal, this simplifies to

$$\Gamma^r_{\phi\phi} = \frac{1}{2} \left(\partial_\phi g_{r\phi} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) = -\frac{1}{2} \partial_r g_{\phi\phi} . \tag{12.16}$$

- We have $g_{\phi\phi} = r^2$ so

$$\Gamma^r_{\phi\phi} = -r \tag{12.17}$$

- Finally, a non-vanishing result!

Polar Coordinates

- It's not *that* hard to calculate the remaining three Christoffels for polar coordinates
- One finds that the only non-vanishing ones are

$$\Gamma^r_{\phi\phi} = -r \,, \quad \Gamma^\phi_{r\phi} = \frac{1}{r} \,. \tag{12.18}$$

- So yes, everyone can do it, but **calculating Christoffels by hand is really tedious**
- Worse, because it's mind-numbingly repetitive, people are prone to make errors for such tasks
- Who is good at performing mind-numbing algorithms for a long time without error? **Computers**
- We will discuss efficient symbolic manipulation packages for GR in the next lecture