Hydrodynamics

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What is Hydrodynamics?

- Formal definition: Hydrodynamics is the effective theory of matter at long-wavelengths
- It's a universal description: applies to *all* systems, regardless of their size, at *long time scales*
- Examples: lakes, earth's atmosphere, glass, rock, the sun, the universe
- A modern formulation of hydrodynamics uses Effective Field Theory (EFT); this is the approach taken in this lecture

Conservation laws

- In lecture 8, we talked about conserved quantities
- An example for a conserved quantity is the electric 4-current

$$j^\mu = \begin{pmatrix} n \\ \vec{j} \end{pmatrix}\,, \tag{15.1}$$

where n is the charge density and $\vec{j}$ is the current density

- The conservation law takes the form

$$\partial_\mu j^\mu = 0\,. \tag{15.2}$$

- You can see it's a conservation law by integrating wrt to volume V

$$\partial_t \int dV\, n = -\int dV \vec{\partial}\cdot\vec{j} \to 0\,. \tag{15.3}$$

- So with $Q = \int dV\, n$ we get $\partial_t Q = 0$

Effective Field Theory

- Suppose we now want to know how an initial distribution of charge evolves in time (dynamics)

$$n(t_0, \mathbf{x}) \to n(t, \mathbf{x}) \tag{15.4}$$

- In full generality, this is a very difficult problem to answer exactly
- But suppose we are not interested to what happens in great detail, we only want to get an idea of where charges end up
- We can use the framework of Effective Field Theory (EFT) for this

A very brief derivation of Hydrodynamics

- In the EFT framework, we have to declare our variables (e.g. the charge density n)
- We then construct unknown properties as gradients of our variables ("gradient expansion")
- For the case at hand, our conservation law in components is

$$\partial_t n + \vec{\partial} \cdot \vec{j} = 0\,. \tag{15.5}$$

- Here n is our variable, so we need to express $\vec{j}$ in terms of n
- To lowest order in gradients, something like $\vec{j} \propto \vec{\partial} n$ works, so we take

$$\vec{j} = -D\vec{\partial} n\,, \tag{15.6}$$

where $D = D(n)$ can be a function of n and the minus sign is convention

A very brief derivation of Hydrodynamics

- For simplicity, we will assume D to be constant (to minimize the writing)
- Plugging

$$\vec{j} = -D\vec{\partial}n\,, \tag{15.7}$$

into the conservation law $\partial_\mu j^\mu = 0$ leads to

$$\partial_t n = D\partial^2 n\,, \tag{15.8}$$

which is known as the **diffusion equation**

- The diffusion equation is an example of *first-order* hydrodynamics, because we only allowed first-order gradients in the construction of $\vec{j}$

A very brief derivation of Hydrodynamics

- Note that in principle, we also could have considered *zeroth-order* diffusion by putting
$$\vec{j} = 0\,. \tag{15.9}$$
- This leads to
$$\partial_t n = 0\,, \tag{15.10}$$
which is not very interesting (constant-in-time charge density)
- I'm only point this out because zeroth-order hydrodynamics happens to be very interesting

Questions on the Diffusion Equation

- Current conservation is exact:

$$\partial_t n + \vec{\partial} \cdot \vec{j} = 0\,. \tag{15.11}$$

- Diffusion equation is

$$\partial_t n = D\partial^2 n\,, \tag{15.12}$$

- Is the diffusion equation exact?
- What system does the diffusion equation describe? Electrons? String Theory? Black Holes?
- How does micro-physics enter the diffusion equation?

Corrections to simple diffusion

There are two main types of corrections to simple diffusion

- Higher order terms in the constitutive relation, e.g.

$$\vec{j} = -D\vec{\partial}n + \alpha\vec{\partial}\partial^2 n + \ldots \tag{15.13}$$

- Non-vanishing sources. For moving charges, there are $\vec{E}, \vec{B}$ fields, so we expect corrections of the form

$$\vec{j} = -D\vec{\partial}n + \sigma\vec{E} + \ldots \tag{15.14}$$

Plugging in the latter into $\partial_\mu j^\mu$ leads to

$$\partial_t n - D\partial^2 n + \sigma\vec{\partial}\cdot\vec{E} = 0\,. \tag{15.15}$$

Diffusion equation from EFT

- For hydro EFT, need good variables (e.g. charge density n)
- Build constitutive relation for $\vec{j}$ from gradients of EFT variables and associated sources (for electromagnetism, electric and magnetic fields)
- Plug in to conservation law $\partial_\mu j^\mu = 0$ to get equation for dynamics
- Only relies on symmetries, no assumption about particular system (universal)
- **Form of equation** is universal, but **coefficients** (e.g. D) depend on micro-physics
- For more details, see for instance my book on hydrodynamics

Hydrodynamics as an EFT

- Let us now consider another conservation law, namely energy-momentum conservation
- This conservation law is expressed through the energy-momentum tensor $T^{\mu\nu}$:

$$\partial_\mu T^{\mu\nu} = 0 \,. \qquad (15.16)$$

- Let us now construct $T^{\mu\nu}$ from an EFT
- EFT variables: a conserved 4-momentum P^μ which fulfills $P^\mu P_\mu = -m^2$
- We can take the EFT variables as energy density ϵ and another 4-vector u^μ normalized such that $u^\mu u_\mu = -1$

Hydrodynamics as an EFT

- Now build $T^{\mu\nu}$ in terms of gradients of ϵ, u^μ
- In addition, we can consider *sources*
- Moving charges create electromagnetic fields
- Moving masses create gravitational fields, so we should supplement metric tensor $g_{\mu\nu}$ (and gradients) as possible source terms for $T^{\mu\nu}$
- Since $T^{\mu\nu}$ is a symmetric rank-two tensor, we have to lowest (zeroth) order in gradients

$$T^{\mu\nu} = c_0(\epsilon) g^{\mu\nu} + c_1(\epsilon) u^\mu u^\nu \,. \tag{15.17}$$

Hydrodynamics as an EFT

- In the *local rest frame* we have $u^\mu = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$
- In Minkowski space-time we then have

$$T^{\mu\nu}_{\rm LRF} = {\rm diag}\left(-c_0 + c_1, c_0, c_0, c_0\right) . \tag{15.18}$$

- The component $T^{00}_{\rm LRF}$ is the Hamiltonian density, so we expect

$$-c_0 + c_1 = \epsilon \tag{15.19}$$

- In addition, one can show that $T^{xx}_{\rm LRF} = p$ is the pressure p of the system related to ϵ via the equation of state $p = p(\epsilon)$
- We thus have $c_0 = p$, $c_1 = \epsilon + P$ such that

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} . \tag{15.20}$$

Hydrodynamics as an EFT

- The result
$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}\,. \qquad (15.21)$$
is the energy-momentum tensor of zeroth-order hydrodynamics (no gradients)
- Its conservation equation in Minkowski space-time
$$\partial_\mu T^{\mu\nu} = 0\,, \qquad (15.22)$$
are four equations, which one can recognize to be the (relativistic) Euler equations of fluid dynamics
- The form of (15.21) is **universal**, e.g. up to higher-order gradient corrections it applies equally well to water and the universe
- Only the equation of state $p = p(\epsilon)$ knows about which system is being described