The covariant derivative

paul.romatschke@colorado.edu

Spring 2021

Review

- In lecture 10, we derived the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\,. \tag{16.1}$$

- We discussed that $\frac{d^2x^\mu}{d\tau^2}$ is not a vector under coordinate transformations
- In this lecture, we will define a new operation that *does* transform properly under coordinate transformations

Vectors and Non-Vectors

- Recall that we are interested in non-linear coordinate transformations

$$dx^\mu \to dx'^\mu = R^\mu_{\ \nu} dx^\nu \,, \tag{16.2}$$

where $R^\mu_{\ \nu}$ does depend on the coordinates

- In the geodesic equation, we have the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$
- We may ask ourselves how these transform under (16.2)
- A somewhat tedious exercise shows that

$$\Gamma'^\mu_{\alpha\beta} = R^\mu_{\ \mu'} (R^{-1})_\alpha^{\ \alpha'} (R^{-1})_\beta^{\ \beta'} \Gamma^{\mu'}_{\alpha'\beta'} - \frac{\partial x'^\mu}{\partial x^{\alpha'} \partial x^{\beta'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} \,. \tag{16.3}$$

- Clearly, $\Gamma^\mu_{\alpha\beta}$ is **not** a tensor under (16.2)

Vectors under General Coordinate Transformations

- If A^μ is a vector under (16.2), $\partial_\nu A^\mu$ will in general **not be a vector** because of coordinate dependence of $R^\mu{}_\nu$
- In more detail, using $A^\mu \to A'^\mu = R^\mu{}_\alpha A^\alpha$ we have

$$\partial_\nu A^\mu \to \partial'_\nu A'^\mu = \partial'_\nu \left(R^\mu{}_\alpha A^\alpha\right) = R^\mu{}_\alpha \partial'_\nu A^\alpha + A^\alpha \partial'_\nu R^\mu{}_\alpha \,. \qquad (16.4)$$

- The second term spoils general covariance of $\partial_\nu A^\mu$

Repairing Covariance

- Very explicitly, using $R^{\mu}{}_{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}$ we have

$$\partial'_{\nu} A'^{\mu} = R^{\mu}{}_{\alpha} \partial'_{\nu} A^{\alpha} + A^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \tag{16.5}$$

- The offending second piece looks a lot like the transformation of the Christoffels, cf. (16.3)
- Let's consider the combination

$$\partial_{\nu} A^{\mu} + \Gamma^{\mu}_{\nu\rho} A^{\rho} \equiv \nabla_{\nu} A^{\mu}\,. \tag{16.6}$$

- Using the transformation properties of both $\Gamma^{\mu}_{\nu\rho}$ and A^{μ}, it is straightforward to show that

$$\nabla_{\nu} A^{\mu} \to \nabla'_{\nu} A'^{\mu} = R^{\mu}_{\alpha} (R^{-1})_{\nu}{}^{\beta} \nabla_{\beta} A^{\alpha}\,. \tag{16.7}$$

The Covariant Derivative

- One can generalize this idea to arbitrary rank tensors
- Defining

$$\begin{aligned}
\nabla_\nu X^{\mu_1\mu_2\ldots\mu_n}{}_{\rho_1\rho_2\ldots\rho_m} &= \partial_\nu X^{\mu_1\mu_2\ldots\mu_n}{}_{\rho_1\rho_2\ldots\rho_m} + \Gamma^{\mu_1}_{\nu\alpha} X^{\alpha\mu_2\ldots\mu_n}{}_{\rho_1\rho_2\ldots\rho_m} \\
&\quad + \Gamma^{\mu_2}_{\nu\alpha} X^{\mu_1\alpha\ldots\mu_n}{}_{\rho_1\rho_2\ldots\rho_m} + \ldots + \Gamma^{\mu_n}_{\nu\alpha} X^{\mu_1\mu_2\ldots\alpha}{}_{\rho_1\rho_2\ldots\rho_m} \\
&\quad - \Gamma^{\beta}_{\nu\rho_1} X^{\mu_1\mu_2\ldots\mu_n}{}_{\beta\rho_2\ldots\rho_m} - \ldots - \Gamma^{\beta}_{\nu\rho_m} X^{\mu_1\mu_2\ldots\mu_n}{}_{\rho_1\rho_2\ldots\beta}
\end{aligned}$$

  these combinations transform as tensors under (16.2)
- We call ∇_μ the (geometric-) covariant derivative and will employ this symbol in order to distinguish it from the usual derivative (∂_μ)

Examples of The Covariant Derivative

Examples for the covariant derivative include

- Scalars:

$$\nabla_\nu X = \partial_\nu X \,. \tag{16.8}$$

- Vectors:

$$\nabla_\nu X^\mu = \partial_\nu X^\mu + \Gamma^\mu_{\nu\alpha} X^\alpha \,. \tag{16.9}$$

- Rank 2 tensors:

$$\nabla_\nu X^{\alpha\beta} = \partial_\nu X^{\alpha\beta} + \Gamma^\alpha_{\nu\rho} X^{\rho\beta} + \Gamma^\beta_{\nu\rho} X^{\alpha\rho} \tag{16.10}$$

Importance of Covariant Derivative

- Covariant derivative generalizes “flat-space” identities
- For instance, $\partial_\mu j^\mu = 0$ is **not** covariant under (16.2)
- Instead of $\partial_\mu j^\mu = 0$, the correct current conservation law is

$$\nabla_\mu j^\mu = 0 \,. \tag{16.11}$$

- This has important consequences. For instance, on a curved manifold

$$\partial_\mu j^\mu = -\Gamma^\mu_{\mu\alpha} j^\alpha \neq 0 \tag{16.12}$$

(sort of like Coriolis force: not “real force”, but you’ll feel it anyway!)

Importance of Covariant Derivative

- The same holds true for energy-momentum conservation. We have

$$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\alpha} T^{\mu\alpha} \,, \tag{16.13}$$

  which in general implies $\partial_\mu T^{\mu\nu} \neq 0$
- Specifically

$$\partial_\mu T^{\mu\nu} = -\Gamma^\mu_{\mu\alpha} T^{\alpha\nu} - \Gamma^\nu_{\mu\alpha} T^{\mu\alpha} \,, \tag{16.14}$$

  so our notion of energy-conservation as in Minkowski space-time no longer holds in GR
- Defining "energy" can become tricky in dynamical situations involving gravity!