Ideal Fluids

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Spring 2021

Review

- In lecture 15, we derived the hydrodynamic energy-momentum tensor

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}\,. \tag{17.1}$$

- In the last lecture, we said that on a curved manifold, it is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{17.2}$$

- Let us now investigate what this conservation law means physically

Eigenvectors

- The 4-vector u^μ is the time-like eigenvector of $T^{\mu\nu}$:

$$u_\mu T^{\mu\nu} = (\epsilon + p) u_\mu u^\mu u^\nu + p u^\nu = \left[(\epsilon + p) \underbrace{u_\mu u^\mu}_{-1} + p \right] u^\nu = -\epsilon u^\nu \tag{17.3}$$

- ϵ is the time-like eigenvalue of $T^{\mu\nu}$
-

$$u_\mu T^{\mu\nu} u_\nu = -\epsilon u^\nu u_\nu = \epsilon \,. \tag{17.4}$$

Projectors

- We can use u^μ to project out time-like properties
- The "rest" is space-like
- Can also write down a space-like projector:

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{17.5}$$

- In the LRF where $u^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$u^\mu u^\nu \to \mathrm{diag}(1,0,0,0)\,, \quad \Delta^{\mu\nu} \to \mathrm{diag}(0,1,1,1)\,. \tag{17.6}$$

Fluid Dynamics

- There are 4 equations in (17.2)
- Let's look at time-like equation:

$$
\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu} &= 0\,, \qquad (17.7)\\
u_\nu \left[u^\mu u^\nu \nabla_\mu (\epsilon + p) + (\epsilon + p) u^\nu \nabla_\mu u^\mu \right. & \\
\left. + (\epsilon + p) u^\mu \nabla_\mu u^\nu + g^{\mu\nu} \nabla_\mu p + p \nabla_\mu g^{\mu\nu} \right] &= 0\,.
\end{aligned}
$$

Fluid Dynamics

- There are 4 equations in (17.2)
- Let's look at time-like equation:

$$
\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu} &= 0\,, \qquad (17.8)\\
u_\nu \Big[u^\mu u^\nu \nabla_\mu(\epsilon + p) + (\epsilon + p) u^\nu \nabla_\mu u^\mu & \\
+ (\epsilon + p) u^\mu \nabla_\mu u^\nu + g^{\mu\nu}\nabla_\mu p + p \underbrace{\nabla_\mu g^{\mu\nu}}_{=0} \Big] &= 0\,.
\end{aligned}
$$

Fluid Dynamics

- There are 4 equations in (17.2)
- Let's look at time-like equation:

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu} &= 0\,, \qquad (17.9)\\ -u^\mu \nabla_\mu(\epsilon + p) - (\epsilon + p)\nabla_\mu u^\mu & \\ +(\epsilon + p) u^\mu u_\nu \nabla_\mu u^\nu + u^\mu \nabla_\mu p &= 0\,. \end{aligned}$$

- Now note

$$u_\nu \nabla_\mu u^\nu = \nabla_\mu \frac{u_\nu u^\nu}{2} = -\nabla_\mu \frac{1}{2} = 0 \qquad (17.10)$$

- So we have

$$u^\mu \nabla_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu = 0\,. \qquad (17.11)$$

Fluid Dynamics

- Time-like part of $\nabla_\mu T^{\mu\nu} = 0$ is

$$u^\mu \nabla_\mu \epsilon + (\epsilon + p) \nabla_\mu u^\mu = 0\,. \qquad (17.12)$$

- Consider flat-space limit: $\nabla_\mu \to \partial_\mu$:

$$u^\mu \partial_\mu \epsilon + (\epsilon + p) \partial_\mu u^\mu = 0 \qquad (17.13)$$

- Consider non-relativistic limit: $u^\mu \to \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$ with $|\vec{v}| \ll 1$:

$$\partial_t \epsilon + \vec{v} \cdot \vec{\partial} \epsilon + (\epsilon + p)\, \vec{\partial} \cdot \vec{v} = 0 \qquad (17.14)$$

- Energy density much bigger than pressure (because of $E = mc^2$):

$$\partial_t \epsilon + \vec{v} \cdot \vec{\partial} \epsilon + \epsilon\, \vec{\partial} \cdot \vec{v} = 0\,, \quad \text{Continuity Equation} \qquad (17.15)$$

Fluid Dynamics

- What about the space-like parts of $\nabla_\mu T^{\mu\nu} = 0$?
- Use space-like projector:

$$\Delta^\alpha_\nu \nabla_\mu T^{\mu\nu} = 0\,. \tag{17.16}$$

- Can again consider flat-space and non-relativistic limit, as well as $\epsilon \gg p$
- Homework: prove that (17.16) in this limit becomes

$$\epsilon \partial_t \vec{v} + \epsilon(\vec{v}\cdot\vec{\partial})\vec{v} + \vec{\partial} p = 0\,, \quad \text{Euler Equation} \tag{17.17}$$

Fluid Dynamics

- We found that $\nabla_\mu T^{\mu\nu} = 0$ corresponds to the combination of continuity and Euler equations in the non-relativistic limit when using the hydrodynamic energy-momentum tensor (17.1)
- In general, $\nabla_\mu T^{\mu\nu} = 0$ with (17.1) are the equations of ideal fluid dynamics in curved space-time
- Here u^μ can be identified with the fluid 4-velocity, and ϵ, p with the fluid's energy density and pressure, related by $p = p(\epsilon)$
- These are *ideal* fluids because we have neglected correction terms to $T^{\mu\nu}$ in our derivation, i.e. gradients
- It is not hard to show that including first-order gradient corrections to $T^{\mu\nu}$, $\nabla_\mu T^{\mu\nu} = 0$ provides the Navier-Stokes equations for a fluid in curved space-time