Einstein Field Equations I

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- We know the energy-momentum tensor is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{18.1}$$

- We also know the metric is covariantly conserved

$$\nabla_\mu g_{\alpha\beta} = 0\,. \tag{18.2}$$

- This begs the question: **Are there any other (rank 2) tensors that vanish under ∇_μ?**
- The answer is affirmative, but it is hard work to find the explicit result. We perform this hard task in this lecture.

Bianchi Identity

- Introduce the commutator of two derivatives as

$$[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \,. \tag{18.3}$$

- Now note

$$[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] V_\sigma + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] V_\sigma + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] V_\sigma = 0 \,, \tag{18.4}$$

because

$$\begin{aligned} &\underbrace{\mu\nu\lambda}_{1} - \underbrace{\mu\lambda\nu}_{2} - \underbrace{\nu\lambda\mu}_{3} + \lambda\nu\mu + \underbrace{\nu\lambda\mu}_{3} - \underbrace{\nu\mu\lambda}_{4} \\ -&\underbrace{\lambda\mu\nu}_{5} + \underbrace{\mu\lambda\nu}_{2} + \underbrace{\lambda\mu\nu}_{5} - \lambda\nu\mu - \underbrace{\mu\nu\lambda}_{1} + \underbrace{\nu\mu\lambda}_{4} = 0 \,. \end{aligned} \tag{18.5}$$

- Eq. (18.4) is known as Bianchi identity

Another identity

Next write

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \,, \\
&= \nabla_\mu \partial_\nu V^\rho + \nabla_\mu \Gamma^\rho_{\nu\alpha} V^\alpha - \nabla_\nu \partial_\mu V^\rho - \nabla_\nu \Gamma^\rho_{\mu\alpha} V^\alpha \\
&= \partial_\mu \partial_\nu V^\rho - \Gamma^\alpha_{\mu\nu} \partial_\alpha V^\rho + \Gamma^\rho_{\mu\alpha} \partial_\nu V^\alpha + \ldots \\
&= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma \\
&= R^\rho{}_{\sigma\mu\nu} V^\sigma \,. \qquad (18.6)
\end{aligned}
$$

Here we have introduced the notation

$$
R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \,. \qquad (18.7)
$$

Bianchi Idendity cont'

- Using this result in the Bianchi idendity 18.4 gives

$$\begin{aligned} &\nabla_\mu R^\rho{}_{\sigma\nu\lambda} V^\sigma - [\nabla_\nu, \nabla_\lambda] \nabla_\mu V^\rho \\ &+\nabla_\nu R^\rho{}_{\sigma\lambda\mu} V^\sigma - [\nabla_\lambda, \nabla_\mu] \nabla_\nu V^\rho \\ &+\nabla_\lambda R^\rho{}_{\sigma\mu\nu} V^\sigma - [\nabla_\mu, \nabla_\nu] \nabla_\lambda V^\rho \quad = \quad 0\,. \end{aligned} \qquad (18.8)$$

- Now write

$$\begin{aligned} [\nabla_\nu, \nabla_\lambda] \nabla_\mu V^\rho &= R^\sigma{}_{\rho\nu\lambda} \nabla_\mu V^\sigma + R_\mu{}^\rho{}_{\nu\lambda} \nabla_\rho V^\sigma\,, \\ &= R^\sigma{}_{\rho\nu\lambda} \nabla_\mu V^\sigma - R^\rho{}_{\mu\nu\lambda} \nabla_\rho V^\sigma\,, \end{aligned} \qquad (18.9)$$

as well as

$$\nabla_\mu R^\rho{}_{\sigma\nu\lambda} V^\sigma = V^\sigma \nabla_\mu R^\rho{}_{\sigma\nu\lambda} + R^\rho{}_{\sigma\nu\lambda} \nabla_\mu V^\sigma \qquad (18.10)$$

Biachi Idendity cont'

- Using this result the Bianchi identity becomes

$$V^\sigma \left(\nabla_\mu R^\rho{}_{\sigma\nu\lambda} + \nabla_\nu R^\rho{}_{\sigma\lambda\mu} + \nabla_\lambda R^\rho{}_{\sigma\mu\nu}\right) + \left(R^\sigma{}_{\mu\nu\lambda} + R^\sigma{}_{\nu\lambda\mu} + R^\sigma{}_{\lambda\mu\nu}\right)\nabla_\sigma V^\rho = 0\,. \quad (18.11)$$

- Using (18.7) one can show (exercise!)

$$R^\sigma{}_{\mu\nu\lambda} + R^\sigma{}_{\nu\lambda\mu} + R^\sigma{}_{\lambda\mu\nu} = 0\,. \quad (18.12)$$

- Since V^σ is arbitary, this means the Bianchi identity becomes

$$\nabla_\mu R^\rho{}_{\sigma\nu\lambda} + \nabla_\nu R^\rho{}_{\sigma\lambda\mu} + \nabla_\lambda R^\rho{}_{\sigma\mu\nu} = 0\,. \quad (18.13)$$

Bianchi Idendity cont'

- Contract with metric tensor and using $\nabla_\mu g^{\alpha\beta}=0$

$$g^{\sigma\nu}\left(\nabla_\mu R^\rho{}_{\sigma\nu\lambda}+\nabla_\nu R^\rho{}_{\sigma\lambda\mu}+\nabla_\lambda R^\rho{}_{\sigma\mu\nu}\right) = 0\,, \tag{18.14}$$

$$\nabla_\mu R^{\rho\nu}{}_{\nu\lambda}+\nabla^\sigma R^\rho{}_{\sigma\lambda\mu}+\nabla_\lambda R^{\rho\nu}{}_{\mu\nu} = 0\,. \tag{18.15}$$

- Contracting again with g^μ_ρ and using $R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}$ gives

$$-\nabla_\mu R^{\nu\mu}{}_{\nu\lambda}-\nabla^\sigma R^\mu{}_{\sigma\mu\lambda}+\nabla_\lambda R^{\mu\nu}{}_{\mu\nu}=0\,. \tag{18.16}$$

Define $R^\rho{}_{\mu\rho\nu}\equiv R_{\mu\nu}$ and $R^\mu_\mu\equiv R$ to find

$$0=-\nabla_\mu R^\mu{}_\lambda-\nabla^\sigma R_{\sigma\lambda}+\nabla_\lambda R=-\nabla_\mu\left(2R^\mu{}_\lambda-g^\mu_\lambda R\right)\,. \tag{18.17}$$

Bianchi Idendity cont'

- After the dust settles, we find

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \,, \tag{18.18}$$

  which is the sought-after third identity that has vanishing covariant derivative.
- We will use this hard-won result in the next lecture to build the Einstein field equations