Einstein Field Equations II

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Review

- We now have three known rank-2 tensors that are covariantly conserved
- These are

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= 0\,. \\ \nabla_\mu g_{\alpha\beta} &= 0\,. \\ \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) &= 0\,. \end{aligned} \tag{19.1}$$

- Based on EFT arguments, these are the only rank-2 tensors with this property
- Let's now study the consequences of (19.1)

Combining Conservation Laws

- The three conservation laws (19.1) can be combined into a single conservation law
- To keep things as general as possible, we allow arbitrary linear combinations of (19.1)
- Choosing two constants Λ, c_2, we get

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} - c_2 T^{\mu\nu} \right) = 0 \, . \tag{19.2}$$

- Let's now look for solutions to the conservation law (19.2)

Einstein Field Equations

- The simplest way to satisfy (19.2) is if the term in parenthesis itself is vanishing
- This leads to the requirement

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = c_2 T^{\mu\nu}\,. \tag{19.3}$$

- Let's study (19.3)!

Some terminology

- We have derived (19.2) from Bianchi identity in an earlier lecture
- Let's give a name to the symbols appearing in (19.2), (19.3)
- We refer to R as the **Ricci scalar**, because it is a scalar quantity (no indices)
- We refer to $R^{\mu\nu}$ as the **Ricci tensor**, because it has two indices
- We got both of these as contractions of $R^{\rho}{}_{\mu\nu\lambda}$, which we refer to as the **Riemann tensor**
- Note that these quantities are actual tensors under general coordinate transformations

Einstein Field Equations

- Back to (19.3)
- (19.3) has many independent components; to get started, let's look at trace
- For the moment, consider **arbitrary** number of space-time dimensions D
- Using $g_{\mu\nu}R^{\mu\nu} = R$ and $g_{\mu\nu}g^{\mu\nu} = D$ we have

$$R - \frac{D}{2}R + D\Lambda = c_2 T_\mu^\mu \qquad (19.4)$$

or

$$R = \frac{D\Lambda - c_2 T_\mu^\mu}{\frac{D}{2} - 1} \, . \qquad (19.5)$$

- For $D = 4$, we have $R = 4\Lambda - c_2 T_\mu^\mu$
- For $D = 2$, there is a problem with our equations (19.3)!

Ricci scalar

- The trace of (19.3) involves the energy-momentum tensor and the Ricci scalar R
- For any manifold, the Ricci scalar is defined in terms of derivatives of the metric tensor $g_{\mu\nu}$
- We can calculate R for manifolds we already encountered before!
- Since I am lazy, I will do this with the help of a symbolic manipulation program

Euclidean coordinates

- Let's start with $\mathbb{R}^N$
- In this case, the metric is $g_{\mu\nu} = \mathrm{diag}(1, 1, 1, \ldots)$
- The Ricci tensor is based on derivatives of $g_{\mu\nu}$, so

$$R = 0\,, \tag{19.6}$$

for $\mathbb{R}^N$

Spherical coordinates

- Let's continue with $\mathbb{R}^3$, but use non-trivial coordinates such as spherical coordinates

$$\vec{x} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} . \tag{19.7}$$

- In terms of $\vec{x}' = (r, \theta, \phi)$, the metric is

$$g'_{\mu\nu} = \mathrm{diag}(1, r^2, r^2\sin^2\theta) . \tag{19.8}$$

- Using the computer to calculate R for this metric, I find

$$R = 0 . \tag{19.9}$$

Spherical surface

- Instead of $\mathbb{R}^3$, let's now look at a spherical *surface*
- The metric is the same as (19.8) but without the radial component, thus

$$g'_{\mu\nu} = r^2 \mathrm{diag}(1, \sin^2\theta)\,, \tag{19.10}$$

  where $r = \mathrm{const.}$

- Using the computer to calculate R for this metric, I find

$$R = \frac{2}{r^2}\,. \tag{19.11}$$

- Non-zero result!

Ricci Scalar

- For $\mathbb{R}^N$, we found $R = 0$ both for Euclidean and spherical coordinates
- It does not matter which coordinates are chosen, R remains the same
- For a spherical surface, we found $R \neq 0$
- R is sensitive to the local *curvature* of the manifold, independent from the coordinates chosen