Weak Departures from Minkowski Space

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Review

- Recall the geodesic equation from lectures 10,11:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{20.1}$$

- In this lecture, we will the geodesic equation in the non-relativistic ("Newtonian") limit
- This means situations where the metric $g_{\mu\nu}$ is *almost* Minkowski.
- AND speeds are non-relativistic

Geodesic Equation

- Let us first rewrite the geodesic equation noting that

$$\frac{dx^\alpha}{d\tau} = u^\alpha\,, \tag{20.2}$$

e.g. a 4-velocity

- Any 4-velocity can be written as

$$u^\alpha = \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}\,. \tag{20.3}$$

where $\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$ is the Lorentz-gamma factor

- If the velocity is much smaller than the speed of light, then

$$\lim_{v \ll 1} u^\alpha \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{20.4}$$

Geodesic Equation

- Since then only $\frac{dx^0}{d\tau} \neq 0$, the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0\,. \tag{20.5}$$

- Next, approximate metric to be near Minkowski:

$$g_{\mu\nu}(t,\vec{x}) = \mathrm{diag}\,(-1,1,1,1) + h_{\mu\nu}(\vec{x})\,, \tag{20.6}$$

  where $h_{\mu\nu} \ll 1$ is *static* (t-indep)

- Calculate Γ^μ_{00} for (20.6) using computer:

$$\Gamma^\mu_{00} = -\frac{1}{2}\begin{pmatrix} 0 \\ \vec{\partial} h_{00} \end{pmatrix}\,. \tag{20.7}$$

Geodesic Equation

- In the NR and near-Minkowski limit, we get for $\mu = i, 0$

$$\frac{d^2\vec{x}}{d\tau^2} - \frac{1}{2}\vec{\partial} h_{00} \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0\,, \quad \frac{d^2 x^0}{d\tau^2} = 0\,. \tag{20.8}$$

- Rewrite this as

$$\frac{d^2\vec{x}}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\vec{x}}{d\tau}\right) = \frac{dx^0}{d\tau}\frac{d}{dx_0}\left(\frac{d\vec{x}}{d\tau}\right) = \frac{dx^0}{d\tau}\frac{d}{dx^0}\left(\frac{dx^0}{d\tau}\frac{d\vec{x}}{dx^0}\right) \tag{20.9}$$

- Now use $\frac{dx^0}{d\tau} = \mathrm{const}$ to find

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\partial} h_{00}\,, \tag{20.10}$$

where $t = x^0$

Geodesic Equation

- The equation
$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\partial} h_{00}\,, \tag{20.11}$$
says acceleration equals gradient of something
- Look familiar?
- Newton's law:
$$\frac{d^2\vec{x}}{dt^2} = -\vec{\partial}\Phi\,, \tag{20.12}$$
where Φ is gravitational potential
- We find that geodesic equation gives Newton's law if
$$\Phi = -\frac{1}{2}h_{00}\,, \quad \text{or} \quad g_{00} = -(1+2\Phi)\,, \tag{20.13}$$
in the weak field limit

Summary

- We started with the geodesic equation and considered weak departures from Minkowski space
- We found that the geodesic equation turns into Newton's law if we identified the metric component g_{00} and Newton's gravitational potential Φ through (20.13)
- We will explore further consequences of this limit in the next lecture