Weak Departures from Minkowski Space II

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Review

- Recall the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = c_2 T_{\mu\nu}\,, \tag{21.1}$$

from the Bianchi identity in lecture 19

- In the last lecture, we considered the weak-field limit of the geodesic equation
- We found that the geodesic equation give the equations of motion for a particle in Newtonian gravity in this limit
- In this lecture, we will explore the Newtonian limit of (21.1)

Weak field limit

- Recall that we are interested in the weak field limit where

$$g_{\mu\nu}(t,\vec{x}) = \mathrm{diag}\,(-1,1,1,1) + h_{\mu\nu}(\vec{x})\,, \tag{21.2}$$

- AND we have a static situation where $h_{\mu\nu}$ does not depend on time
- Because it's static, we also have that fluid 4-velocities are

$$\lim_{v\ll 1} u^\alpha \to \left(\begin{array}{c} 1 \\ 0 \end{array}\right)\,. \tag{21.3}$$

- In addition, we are assuming that the pressure is much smaller than the energy density (because the latter contains the rest mass energy)

$$p \ll \epsilon \tag{21.4}$$

Einstein Field Equations

- Field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = c_2 T_{\mu\nu}\,, \tag{21.5}$$

- Trace of field equations is

$$R = 4\Lambda - c_2 T^{\mu}_{\mu} \tag{21.6}$$

- Use in field equations to get

$$R_{\mu\nu} = c_2 T_{\mu\nu} + g_{\mu\nu} \left(\Lambda - \frac{c_2}{2} T^{\alpha}_{\alpha}\right)\,. \tag{21.7}$$

- Look at 00 components in weak-field limit!

Einstein Field Equations

- Metric tensor in weak-field limit:

$$g_{00} = -1 + h_{00} = -1 - 2\Phi \tag{21.8}$$

- Ricci tensor (computer!):

$$R_{00} = -\frac{1}{2}\vec{\nabla}^2 h_{00} = \vec{\nabla}^2 \Phi \,. \tag{21.9}$$

- Energy-momentum tensor:

$$T_{00} = (\epsilon + p) u_0^2 + p g_{00} \,. \tag{21.10}$$

Energy-Momentum tensor

- Note: $u^\mu u_\mu = -1 = u^\mu u^\nu g_{\mu\nu}$
- Static means $\vec{u} = 0$ so

$$-1 = (u^0)^2 g_{00} = g^{00}(u_0)^2 \,, \tag{21.11}$$

- Now in the weak-field limit $\Phi \ll 1$

$$g^{00} = \frac{1}{g_{00}} \tag{21.12}$$

- We get

$$u_0^2 = \frac{-1}{g^{00}} = -g_{00} = 1 + 2\Phi \tag{21.13}$$

- So

$$u_0 = 1 + \Phi \tag{21.14}$$

Energy-Momentum tensor

- Energy-momentum tensor

$$T_{00} = (\epsilon + p) u_0^2 + p g_{00} \tag{21.15}$$

- Use $g_{00} = -1 - 2\Phi$ and $u_0 = 1 + \Phi$
- Find

$$T_{00} = \epsilon (1 + 2\Phi) \,. \tag{21.16}$$

- In addition

$$T_\mu^\mu = -\epsilon + 3p \tag{21.17}$$

and in the non-relativistic limit $p \ll \epsilon$ so

$$T_\mu^\mu = -\epsilon \tag{21.18}$$

Field Equations

- Putting everything together, the field equations (21.1) become

$$\vec{\partial}^2\Phi = (1+2\Phi)\left[\frac{c_2}{2}\epsilon - \Lambda\right]\,. \tag{21.19}$$

- Φ on the rhs is much smaller than unity, so

$$\vec{\partial}^2\Phi = \left[\frac{c_2}{2}\epsilon - \Lambda\right]\,. \tag{21.20}$$

- Next, **assume** $\Lambda \ll \epsilon$ (for no good reason right now)

$$\vec{\partial}^2\Phi = \frac{c_2}{2}\epsilon\,. \tag{21.21}$$

Einstein Field Equations

- We find the field equations (21.1) in the weak-field non-relativistic limit give

$$\vec{\partial}^2\Phi = \frac{c_2}{2}\epsilon\,. \tag{21.22}$$

- Let's compare to Newton's law of gravity:

$$\vec{\partial}^2\Phi = 4\pi G\epsilon\,. \tag{21.23}$$

- The two match **if**

$$c_2 = 8\pi G \tag{21.24}$$

Einstein Field Equations

- Choosing $c_2 = 8\pi G$, the field equations (21.1) are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{21.25}$$

- These correspond to Newton's law of gravity in the weak-field limit
- Away from the weak-field limit, (21.25) are field equations for gravity that generalize Newton's equations
- Eq. (21.25) are the **Einstein Field Equations**