Schwarzschild Solutions I

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Review

- The last 4 lectures were dedicated to deriving the Einstein Field Equations
- These are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \qquad (22.1)$$

- In the last lecture, we found that (22.1) correspond to Newton's law of gravity in the weak-field limit
- In this lecture, we want to explore solutions to (22.1) outside the Newtonian limit

Empty Space

- Eq. (22.1) constitute non-linear PDE's for the metric $g_{\mu\nu}$ with source $T_{\mu\nu}$
- Let's simplify our task by considering **emtpy space**:

$$T_{\mu\nu} = 0\,. \tag{22.2}$$

- This means we study *pure gravity*, no matter at all
- For empty space:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0\,. \tag{22.3}$$

No Cosmological Constant

- In addition to empty space, there is the constant Λ, which –following Einstein – we call **Cosmological Constant**
- It will become important later on, but for now, let's assume

$$\Lambda = 0\,. \tag{22.4}$$

- We will find that this is a good approximation for our current goal
- For empty space and vanishing cosmological constant, we have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0\,. \tag{22.5}$$

Trace

- Taking the trace of the field equations, we have

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = 0 = R - 2R\,. \tag{22.6}$$

  so $R = 0$

- Therefore, for empty space and vanishing Λ, (22.1) become

$$R_{\mu\nu} = 0\,. \tag{22.7}$$

- Let's look for solutions of (22.7) now

Symmetric Solutions

- Eq. (22.7) is still complicated non-linear PDE
- Let's simplify further by demanding *symmetries*
- For instance, we can look for *static* solutions, e.g. $g_{\mu\nu} = g_{\mu\nu}(\vec{x})$
- In addition, we can look for *spherically symmetric* solutions, e.g. $g_{\mu\nu} = g_{\mu\nu}(|\vec{x}|)$
- For spherically symmetric problems, it is useful to employ spherical coordinates r, θ, ϕ

Spherical Symmetry

- For spherically symmetric problems, it is useful to employ spherical coordinates r, θ, ϕ
- The Minkowski metric in spherical coordinates is

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2\theta)$$

- The Minkowski line element in spherical coordinates thus is

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu} = -dt^2 + dr^2 + r^2 d\Omega^2\,, \qquad (22.8)$$

  where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
- Let's make an **ansatz** for ds^2 for static & spherically symmetric manifolds:

$$ds^2 = \ -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2\,. \qquad (22.9)$$

Spherical Symmetry

- For our ansatz (22.9), the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} . \tag{22.10}$$

- Here A, B are unknown functions of the radius
- Let's try to see if our ansatz can be used to solve the field equations (22.7)

Spherical Symmetry

- We need the Ricci tensor $R_{\mu\nu}$ for the metric (22.10)
- We use symbolic manipulation to calculate $R_{\mu\nu}$ on the computer
- Using the explicit form of $R_{\mu\nu}$ we find

$$B(r)R_{00} + A(r)R_{11} = \frac{BA' + AB'}{rB} = 0\,, \qquad (22.11)$$

  where the rhs is a consequence of the field equations (22.7) .
- The other two Einstein equations $R_{22} = 0$ and $R_{33} = 0$ lead to the same equation:

$$2AB(B-1) - r(BA' - AB') = 0\,. \qquad (22.12)$$

Spherical Symmetry

- We find that for the ansatz (22.9), the combination (22.11) is solved if
$$BA' + AB' = 0 \qquad \text{or} \quad B(r) = \frac{1}{A(r)} \tag{22.13}$$
- Plugging this into (22.12) leads to
$$-1 + \frac{1}{A} - \frac{rA'}{A} = 0 \tag{22.14}$$
- This is an ODE and it has the solution
$$A(r) = 1 + \frac{\text{const}}{r}\,. \tag{22.15}$$
- We check explicity that this solution for A together with $B = \frac{1}{A}$ solves
$$R_{\mu\nu} = 0\,. \tag{22.16}$$

Schwarzschild Solution

Karl Schwarzschild

- This solution to the Einstein Field Equations was found by Schwarzschild in 1915
- Wrote the paper same year Einstein published GR
- Wrote the paper during WWI
- Wrote the paper while fighting on the Russian front
- Died in 1916