Schwarzschild Solutions II

paul.romatschke@colorado.edu

Spring 2021

Review

- Einstein Field Equations are hard to solve in general
- In the last lecture, we discussed an analytical solution found by Schwarzschild
- The solution in spherical coordinates is given by

$$g_{\mu\nu} = \mathrm{diag}\left(-A(r), A^{-1}(r), r^2, r^2 \sin^2\theta\right) \,, \tag{23.1}$$

where $A(r)$ is a function that is given by

$$A(r) = 1 + \frac{\mathrm{const}}{r} \tag{23.2}$$

- In this lecture, we will discuss the physical meaning of the Schwarzschild solution

The Schwarzschild Radius

- Let's focus on the constant first.
- Writing $\mathrm{const} = -r_s$ we have

$$A = 1 - \frac{r_s}{r}\,, \tag{23.3}$$

  where r_s is an unknown constant of dimension length
- We call r_s the **Schwarzschild radius**
- Let's try to relate r_s to things we know

The Schwarzschild Radius

- For the Schwarzschild solution

$$g_{00} = -A = -1 + \frac{r_s}{r}\,, \tag{23.4}$$

- For large radius $r \to \infty$, g_{00} is very close to Minkowski
- Therefore, for $r \to \infty$ we are in the *weak field limit* where

$$-1 + \frac{r_s}{r} = g_{00} = -(1 + 2\Phi)\,, \qquad \vec{\partial}^2\Phi = 4\pi G\epsilon \tag{23.5}$$

  and Φ the Newton gravitational potential
- For spherical symmetry, Newton's potential is

$$\Phi = -\frac{GM}{r}\,, \tag{23.6}$$

  where $M = \int dV\epsilon$ is the total mass in a ball of radius r

The Schwarzschild Radius

- We find
$$r_s = 2GM\,, \tag{23.7}$$
for the Schwarzschild radius
- Let's look at the metric again:
$$g_{00} = -(1 - \frac{2GM}{r})\,. \tag{23.8}$$
- For $r \to \infty$, $g_{00} \to -1$ is nicely behaved
- Decreasing r, g_{00} is growing
- For $r = r_s$ we have
$$g_{00}(r = r_s) = 0\,, \quad g_{rr}(r = r_s) = \infty \tag{23.9}$$
- The metric is **singular** at $r = r_s$
- We call $r = r_s$ the **horizon**

A second singularity

- Let's look at the metric component once more:

$$g_{00} = -(1 - \frac{2GM}{r}) \,. \tag{23.10}$$

- Besides the singularity at $r = r_s$, there is a **second singularity** at

$$r = 0 \,. \tag{23.11}$$

- The Schwarzschild solution is singular for both $r = 0, r_s$
- What do these singularities mean physically? Are they "real" singularities?

Coordinate singularities

- Let's get some insight from polar coordinates
- For $\vec{x} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$ we have

$$g_{ij} = \mathrm{diag}\left(1, r^2\right) , \quad g^{ij} = \mathrm{diag}\left(1, \frac{1}{r^2}\right) . \tag{23.12}$$

- Obviously, the metric is singular at $r = 0$
- Not a problem of the manifold: $\mathbb{R}^2$ is fine
- $r = 0$ is not a "real" singularity
- It's just a consequence of the polar coordinates we employed
- We call these type of singularities **"coordinate singularities"**

Coordinate vs. Real Singularities

- How do we decide if the singularities $r = 0, r_s$ of the Schwarzschild solution are "real" or "coordinate" singularities?
- Need coordinate-invariant tests!
- Fortunately, we have coordinate-invariant quantities, e.g.

$$R = R^{\mu}_{\mu}\,, \quad R^{\mu\nu}R_{\mu\nu}\,, \quad R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\ \ldots \tag{23.13}$$

Coordinate vs. Real Singularities

- Let's check these for Schwarzschild
- We find
$$R(r) = 0\,, \tag{23.14}$$
so this doesn't really tell us much
- Also $R^{\mu\nu}R_{\mu\nu} = 0$ for all r because Einstein equations are $R_{\mu\nu} = 0$
- Better:
$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{6r_s^2}{r^2} \tag{23.15}$$
- For $r = r_s$, nothing bad happens. So $r = r_s$ is a coordinate-singularity!
- By contrast, for $r = 0$ (23.15) diverges. So $r = 0$ is a real singularity!