Killing Vectors

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Killing Vectors

- We have discussed Noether theorem and conserved charges in Minkowski space in lecture 8
- In this lecture, we will consider the equivalent on **curved Manifolds**
- The conserved quantities are called **Killing Vectors**
- Their derivation is the subject of this lecture

Geodesic Equation

- Consider the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} - \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{24.1}$$

- Now rewrite the geodesic equation for *downstairs* indices

$$\frac{d^2x_\mu}{d\tau^2} - \Gamma^\alpha_{\mu\beta}\frac{dx_\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0\,. \tag{24.2}$$

- Let $u^\alpha \equiv \frac{dx^\alpha}{d\tau}$ so that

$$\frac{du_\mu}{d\tau} = \Gamma^\alpha_{\mu\beta}u_\alpha u^\beta\,. \tag{24.3}$$

Christoffel symbols

- In this equation, replace $\Gamma^{\alpha}_{\mu\beta}$ by its definition in terms of derivatives of the metric tensor
- For notational brevity, I will denote

$$\partial_\mu g_{\alpha\beta} = g_{\alpha\beta,\mu} \tag{24.4}$$

- We get

$$\frac{du_\mu}{d\tau} = \frac{1}{2} g^{\alpha\nu}\left(g_{\nu\beta,\mu} + g_{\nu\mu,\beta} - g_{\mu\beta,\nu}\right) u_\alpha u^\beta\,, \tag{24.5}$$

$$= \frac{1}{2}\left(g_{\nu\beta,\mu} + g_{\nu\mu,\beta} - g_{\mu\beta,\nu}\right) \underbrace{u^\nu u^\beta}_{\text{symm. in } \nu,\beta} \tag{24.6}$$

Christoffel symbols

- In this equation, replace $\Gamma^{\alpha}_{\mu\beta}$ by its definition in terms of derivatives of the metric tensor
- For notational brevity, I will denote

$$\partial_\mu g_{\alpha\beta} = g_{\alpha\beta,\mu} \tag{24.7}$$

- We get

$$\frac{du_\mu}{d\tau} = \frac{1}{2} g^{\alpha\nu} \left(g_{\nu\beta,\mu} + g_{\nu\mu,\beta} - g_{\mu\beta,\nu}\right) u_\alpha u^\beta\,, \tag{24.8}$$

$$= \frac{1}{2}\left(g_{\nu\beta,\mu} + \underbrace{g_{\nu\mu,\beta} - g_{\mu\beta,\nu}}_{\text{anti}-\text{symm. in } \nu,\beta}\right) \underbrace{u^\nu u^\beta}_{\text{symm. in } \nu,\beta} \tag{24.9}$$

Collect & Simplify

- We get
$$\frac{du_\mu}{d\tau} = \frac{1}{2} g_{\nu\beta,\mu} u^\nu u^\beta \tag{24.10}$$

- Now **if** $g_{\nu\beta}$ is **independent from** coordinate x^μ, we have $g_{\nu\beta,\mu} = 0$ and hence
$$\frac{du_\mu}{d\tau} = 0\,. \tag{24.11}$$

- From independence follows that
$$u_\mu = \text{const.} \tag{24.12}$$

- We get something like a conservation law!

Killing Vectors

- Let's do an example
- If t is the coordinate to which $\partial_t g = 0$, the operator ∂_t acts like a symmetry
- We may write $\partial_t = \delta_t^\mu \partial_\mu \equiv K^\mu \partial_\mu$
- Here we have identified the Killing vector

$$K^\mu = \delta_t^\mu \tag{24.13}$$

- In terms of the Killing vector, the conservation law becomes

$$K^\sigma u_\sigma = \mathrm{const} = \delta_t^\sigma u_\sigma = u_t \,. \tag{24.14}$$

  which matches (24.12) for $\mu = t$
- The physical interpretation of this is trivial: if the metric is time-independent, nothing depends on time, and hence the Lorentz factor u_t is constant.

Killing Vectors

- In the present context, it's hard to see the use of Killing vectors
- However, when we use more complicated coordinates, Killing vectors become useful
- We will start to appreciate Killing vectors when we calculate geodesics in the Schwarzschild geometry