Equations of Motion for Schwarzschild

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Review

- We have discussed the Schwarzschild solution to the Einstein Field Equations in lectures 22, 23
- This is a spherically symmetric & static solution
- The solution is for empty space (no matter), except near the center (for $r \geq r_s$)
- Very much like the gravitational field of a planet in space!
- The Schwarzschild solution is the generalization of Newton's gravitational potential for a mass in outer space!

Geodesics for Schwarzschild

- Let's consider a particle in the Schwarzschild spacetime
- If it was Newton gravity, the particle would follow a trajectory in the gravitational field of the planet
- In GR, the particle's trajectory is dictated by the geodesic equation
- So our aim for this lecture is to find trajectories for Schwarzschild, and compare to "orbits" in Newtonian gravity
- This is hard to do directly
- We will use an alternative route for conserved quantities using Killing vectors

Killing Vectors

- The Schwarzschild solution is static and spherically symmetric
- As a consequence, the Schwarzschild metric obeys

$$\partial_t g_{\mu\nu} = 0\,, \quad \partial_\phi g_{\mu\nu} = 0\,. \tag{25.1}$$

- We have at least two Killing vectors (see lecture 24)!
- One can show that there is at least one additional Killing vector
- The Killing vectors are

$$K_t^\mu = \delta_t^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,, \quad K_\theta^\mu = \delta_\theta^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,, \quad K_\phi^\mu = \delta_\phi^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{25.2}$$

Conserved Quantities

- There is a conserved quantity associated with each Killing vector
- For the Schwarzschild solution, these conserved quantities are

$$-K_t^{\mu} u_{\mu} = -K_{t,\mu} u^{\mu} = -K_{t,0} \frac{dt}{d\tau} = E \quad (\text{“Energy”}) \tag{25.3}$$

  (the minus sign is convention)
- Furthermore:

$$K_{\phi,\mu} u^{\mu} = K_{\phi,3} \frac{d\phi}{d\tau} = L \quad (\text{“Angular Momentum”}) \tag{25.4}$$

- In addition, there is a third conserved quantity. For the following discussion, we choose it to be equal to zero:

$$K_{\theta,\mu} u^{\mu} = K_{\theta,2} \frac{d\theta}{d\tau} = 0\,. \tag{25.5}$$

Conserved Quantities

- Lower-index Killing vectors are found from upstairs-index through metric contraction
- To be explicit, we have

$$K_{t,\mu} = g_{\mu\nu} K_t^\mu = g_{\mu\nu} \delta_t^\mu = g_{t\nu} = \begin{pmatrix} -(1-\frac{r_s}{r}) \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{25.6}$$

- Proceeding in a similar fashion, we get

$$K_{\phi,\mu} = g_{\phi\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ r^2 \sin^2\theta \end{pmatrix} . \tag{25.7}$$

Conserved Quantities

- Using these explicit formulas for the Killing vectors, the conserved quantities are

$$E = \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}\,, \quad L = r^2 \sin^2\theta \frac{d\phi}{d\tau}\,. \tag{25.8}$$

- In addition, the third conservation law is

$$\frac{d\theta}{d\tau} = 0\,, \quad \text{or} \quad \theta = \text{const}\,. \tag{25.9}$$

- For simplicity, in the following we choose the constant as

$$\theta = \frac{\pi}{2}\,. \tag{25.10}$$

Massive Particle

- For a massive particle, it's proper time is related to the line element as

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \,. \tag{25.11}$$

- We can rewrite this condition by dividing both sides by $d\tau$ to find

$$-1 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \,. \tag{25.12}$$

- Of course, this is nothing but our normalization condition for the particle's velocity $u^\mu = \frac{dx^\mu}{d\tau}$

$$u^\mu u_\mu = -1 \,. \tag{25.13}$$

Towards equations of motion

- Let's write out this condition in components:

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 \quad (25.14)$$

- Next, use the conserved quantities to express this equation as

$$-1 = g_{tt}\frac{E^2}{g_{tt}^2} + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\frac{L^2}{g_{\phi\phi}^2}\,. \quad (25.15)$$

- For the Schwarzschild solution $g_{rr} = -\frac{1}{g_{rr}} = \frac{1}{A(r)}$ so that

$$-A(r) = -E^2 + \left(\frac{dr}{d\tau}\right)^2 + A(r)\frac{L^2}{g_{\phi\phi}}\,. \quad (25.16)$$

Towards equations of motion

- Next consider $g_{\phi\phi} = r^2 \sin^2\theta$ and recall that we set $\theta = \frac{\pi}{2}$ using the third Killing vector
- This gives

$$-A(r) = -E^2 + \left(\frac{dr}{d\tau}\right)^2 + \frac{A(r)}{r^2} L^2 \,. \tag{25.17}$$

- When plugging in the Schwarzschild solution $A(r) = 1 - \frac{r_s}{r}$ we finally get

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2 \,. \tag{25.18}$$

The equations of motion

- For an arbitrary manifold, the equations of motion for a particle are given by the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} - \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \,. \tag{25.19}$$

- This equation is difficult to solve
- Using instead conserved quantities resulting from the symmetries of the spacetime, we found a much simpler equation of motion

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2 \,, \tag{25.20}$$

where L, E are constants of motion.