Circular closed orbits – Newtonian Theory vs. GR

paul.romatschke@colorado.edu

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Review

- The Schwarzschild solution is a solution to the equations of GR
- We discussed the Schwarzschild solution for a mass in outer space
- We derived an equation of motion for a massive particle for the Schwarzschild solution
- In this lecture, we consider *circular orbits* for a particle
- We will compare circular orbits in GR to Newtonian gravity

Equations of motion

- In lecture 25, we found that a massive particle with energy E and angular momentum L has the GR equations of motion

$$\frac{E^2}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \qquad (26.1)$$

- This is similar to the "energy balance" in classical Newtonian mechanics
- Specifically consider a particle with mass m and velocity v in a potential $V(r)$
- In classical Newtonian mechanics, the particle's energy $\mathcal{E}$ would be given by

$$\mathcal{E} = \frac{mv^2}{2} + mV(r). \qquad (26.2)$$

Equations of motion

- If the particle's velocity is *radial* $v = \frac{dr}{d\tau}$, so the Newtonian theory gives

$$\frac{\mathcal{E}}{m} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r)\,. \tag{26.3}$$

- We can compare this to the GR equation of motion (26.1)

$$\frac{E^2}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{26.4}$$

- We conclude that for GR, there is an "energy-budget" similar to classical mechanics
- The only difference between GR and Newton is the form of the potential $V(r)$

Gravitational Potential

- In Newton's theory of gravity, the gravitational potential away from a body of mass M is $V_N(r) = -\frac{GM}{r}$
- For a particle of angular momentum L, in classical mechanics there is an additional potential term given by $V_L(r) = \frac{L^2}{2r^2}$
- So the potential for a particle in classical (Newtonian) mechanics we have

$$V_{\rm class}(r) = V_N(r) + V_L(r) = -\frac{GM}{r} + \frac{L^2}{2r^2}\,. \tag{26.5}$$

- By contrast, in GR we find from (26.1)

$$V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(1+\frac{L^2}{r^2}\right) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\,. \tag{26.6}$$

Circular Orbits – Newton

- Circular orbits mean radius does not change: $\frac{dr}{d\tau} = 0$
- **Stable** circular orbits are found from the minimum of the potential
- For classical (Newton) mechanics, we have

$$0 = \frac{dV_{\rm class}}{dr} = \frac{GM}{r^2} - \frac{L^2}{r^3} \tag{26.7}$$

- So for Newton gravity, stable circular orbits are found for

$$r = \frac{L^2}{GM}\,. \tag{26.8}$$

- For Newton gravity, orbits are stable for all L

Circular Orbits – Einstein

- Circular orbits mean radius does not change: $\frac{dr}{d\tau} = 0$
- **Stable** circular orbits are found from the minimum of the potential
- For GR we have

$$0 = \frac{dV}{dr} = \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3GML^2}{r^4} \tag{26.9}$$

- So for GR we get

$$r^2 - \frac{L^2}{GM} r + 3L^2 = 0\,, \quad r = \frac{L^2}{2GM} \pm \sqrt{\frac{L^4}{4G^2M^2} - 3L^2}\,. \tag{26.10}$$

Stable circular orbits exist as long as $\frac{L^2}{4G^2M^2} \geq 3$ or

$$|L| \geq \sqrt{12} GM \equiv L_c\,. \tag{26.11}$$

Circular Orbits – Einstein

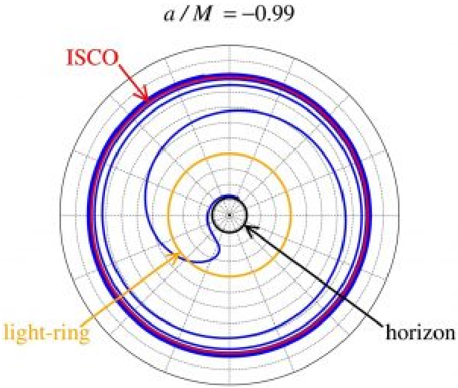
- For GR, there is a minimum angular momentum $L = L_c = \sqrt{12}GM$ below which orbits are no longer stable
- There is an associated minimum radius r_{ISCO}:

$$r_{ISCO} = \frac{L_c^2}{2GM} = 6GM\,. \tag{26.12}$$

- We call this the **Innermost Stable Circular Orbit** (ISCO)
- In relation to the Schwarzschild horizon $r_s = 2GM$ we have

$$r_{ISCO} = 3r_s\,. \tag{26.13}$$

Circular Orbits – Einstein



Sketch of GR orbits