Non-circular closed orbits

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Review

- We discussed closed circular orbits for the Schwarzschild solution
- Let's now generalize this to *non-circular* orbits
- We will compare again compare orbits in GR to Newtonian gravity
- A key application of our calculation is a prediction for perihelon advance – an important observational test of GR!

Equations of motion

- In lecture 25, we found that a massive particle with energy E and angular momentum L has the GR equations of motion

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{27.1}$$

- In lecture 26, we considered circular orbits with $r = \mathrm{const}$ so that $\frac{dr}{d\tau} = 0$
- Let's now look for non-circular orbits with $\frac{dr}{d\tau} \neq 0$
- Recall that we have conserved quantities such as the angular momentum

$$L = r^2 \frac{d\phi}{d\tau}. \tag{27.2}$$

Equations of motion

- In lecture 25, we found that a massive particle with energy E and angular momentum L has the GR equations of motion

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{27.3}$$

- For non-circular orbits, multiply (27.3) by $(\frac{d\tau}{d\phi})^2 = \frac{r^4}{L^2}$

$$E^2 \frac{r^4}{L^2} = \left(\frac{dr}{d\phi}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(r^2 + \frac{r^4}{L^2}\right). \tag{27.4}$$

- Now define the inverse radius as a new variable:

$$u = \frac{1}{r}, \quad du = -\frac{1}{r^2} dr = -u^2 dr \tag{27.5}$$

Equations of motion

- For a non-circular orbit $\frac{du}{d\phi} \neq 0$, so we can divide and get

$$\frac{d^2u}{d\phi^2} + u - \frac{3r_s}{2}u^2 - \frac{r_s}{2L^2} = 0\,. \tag{27.6}$$

- This is a non-linear ODE; cannot solve exactly analytically
- Two choices:
  1. Do approximations
  2. Solve numerically

Equations of motion – Linear Approximation

- Equations of motion:

$$\frac{d^2u}{d\phi^2} + u - \frac{3r_s}{2}u^2 - \frac{r_s}{2L^2} = 0\,. \tag{27.7}$$

- First approximation: neglect nonlinear (u^2) term:

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2}\,. \tag{27.8}$$

- Can solve exactly:

$$u = \frac{r_s}{2L^2} + \alpha\cos(\phi + \beta)\,. \tag{27.9}$$

- It's an ellipse! Keppler's law!

Equations of motion – Beyond Linear Approximation

- Let's now include the non-linear term:

$$\frac{d^2u}{d\phi^2} + u - \frac{3r_s}{2}u^2 - \frac{r_s}{2L^2} = 0\,. \tag{27.10}$$

- We could try looking for solutions which are close to Kepler's law:

$$u = \frac{r_s}{2L^2} + \alpha\cos(\phi + \beta) + \delta u = u_{\rm Keppler} + \delta u\,, \quad \delta u \ll u\,. \tag{27.11}$$

- Plugging (27.11) into (27.10) and linearizing in δu we get

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3r_s}{2}u^2_{\rm Keppler} \tag{27.12}$$

- Solving this is possible, but complicated!

Equations of motion – Beyond Linear Approximation (2)

- Let's now include the non-linear term:

$$\frac{d^2u}{d\phi^2} + u - \epsilon \frac{3r_s}{2} u^2 - \frac{r_s}{2L^2} = 0\,. \tag{27.13}$$

- $\epsilon = 1$, but we will pretend $\epsilon \ll 1$ to get a solution
- Instead of linearizing around $u = u_{\rm Keppler}$, we could try adjusting the Kepplerian frequency
- Specifically, let's do

$$u = C_0 + \alpha \cos\left(\phi C_1 + \beta\right) + \epsilon c_2 \cos\left(2\phi + \beta_2\right)\ , \tag{27.14}$$

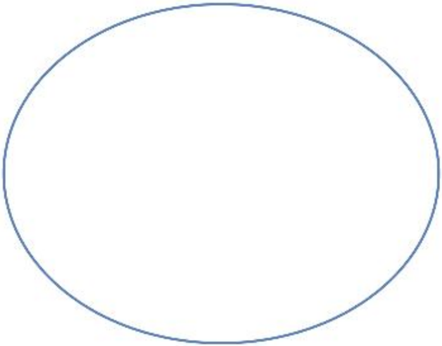
where $C_0 = \frac{r_s}{2L^2} + \epsilon c_0$, $C_1 = 1 + \epsilon c_1$

Equations of motion – Beyond Linear Approximation (2)

- Plugging this into (27.13) and linearizing in $\epsilon \ll 1$, we find a solution with

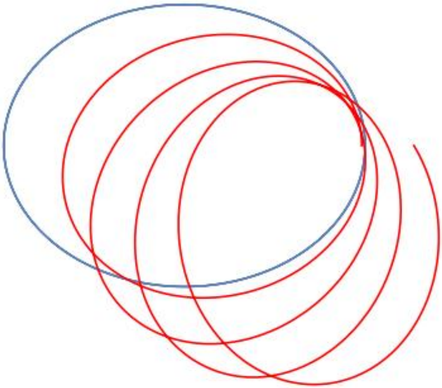
$$c_0 = \frac{3r_s}{8}\left(\frac{r_s^2}{L^2} + 2\alpha^2\right), \quad c_1 = -\frac{3r_s^2}{4L^2}, \quad c_2 = -\frac{r_s\alpha^2}{4}. \qquad (27.15)$$

Orbits – Newton vs. Einstein



Keppler: planetary motion is elliptic

Orbits – Newton vs. Einstein



Einstein: planetary motion is elliptic with moving shape

Getting Quantitative

- The moving shape found in GR is distinct from Newton gravity
- It can be used as a *observational test* for GR
- If GR is the correct theory (rather than Newton gravity), then e.g. planetary motion in our solar system should show this "advance" of the elliptic shape
- We will study quantitative predictions vs. observations for Mercury in the next lecture