Mercury Perihelon Advance

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Review

- In the last lecture, we discussed *non-circular* orbits in GR
- We found that – in contrast to Newton gravity – the solutions predict an evolving elliptic shape
- In this lecture, we will use our results applied to Mercury – and offer a first precision test for GR as a theory of gravity

Non-circular orbits in GR

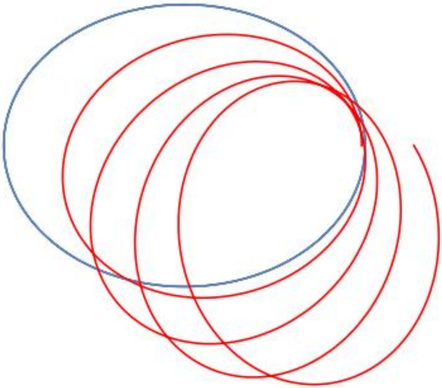
- In the last lecture, we found an (approximate) solution for a non-circular orbit in GR
- In terms of radius as a function of the angle, the relevant part of the solution is

$$r_{\rm GR}(\phi)=\frac{1}{\frac{GM}{L^2}+{\rm const}\times\cos\left(\phi(1-\frac{3r_s^2}{4L^2})\right)}\,, \qquad (28.1)$$

- This is to be contrasted with Newton's theory, which says

$$r_{\rm Newton}(\phi)=\frac{1}{\frac{GM}{L^2}+{\rm const}\times\cos\left(\phi\right)} \qquad (28.2)$$

Orbits – Newton vs. Einstein



Perihelon: the point farthest from the sun

Getting Quantitative

- We can calculate how much perihelon advances:
- $$\text{Newton}: \cos(\phi)\,, \quad \text{GR}: \cos\left(\phi(1-\frac{3r_s^2}{4L^2})\right) \qquad (28.3)$$
- For GR, $\phi = 2\pi$ does not bring us back to the same point
- Rather, for GR the same point *advances* by an angle of

$$\Delta\phi = 2\pi \times \frac{3r_s^2}{4L^2} \qquad (28.4)$$

for *every orbit*

Getting Quantitative

- Since this effect is not present for Newtonian gravity, we can test if GR really describes gravity in nature
- Let's reinstate all constants: perihelon advance per orbit is

$$\Delta\phi = 2\pi \times \frac{3G^2M^2}{L^2} \,. \tag{28.5}$$

- To evaluate $\Delta\phi$ we need GM (for the sun) and L for Mercury

Getting Quantitative

- First recall that for an ellipse with semi-major axis a and eccentricity e the radius is

$$r(\phi) = \frac{(1-e^2)a}{1+e\cos\phi} = \frac{1}{\frac{GM}{L^2} + \mathrm{const} \times \cos\phi}\,. \tag{28.6}$$

  where the second equality comes from our solution (28.1)
- In order for the constants to match up we need

$$(1-e^2)a = \frac{L^2}{GM}\,. \tag{28.7}$$

- We can plug this into (28.5) to get

$$\Delta\phi = 2\pi \times \frac{3GM}{(1-e^2)a}\,. \tag{28.8}$$

Table of Perihelon advance – solar system

- $\Delta\phi$ is largest for smallest orbit/largest eccentricity
- Let's search a, e for planets in our solar system!

Planet	a [Mkm]	e
Mercury	57.9	0.2056
Venus	108.2	0.0067
Earth	149.6	0.0167
Mars	227.9	0.0934
...	...	...

Mercury wins!

Mercury Perihelon Advance

- Let's get quantitative for Mercury's GR perihelon advance
- We have

$$\Delta\phi = 2\pi \times \frac{3GM}{(1-e^2)a}\,. \tag{28.9}$$

- The sun has a mass of $M = 1.99 \times 10^{30}$ kg so

$$\frac{GM}{c^2} \simeq 1476m\,. \tag{28.10}$$

- Plugging in $a = 5.79 \times 10^{10}$m and $e = 0.2056$ for Mercury gives

$$\Delta\phi = 2\pi \times 8 \times 10^{-8}\mathrm{rad} \tag{28.11}$$

per orbit

Mercury Perihelon Advance

- This is a very small angle per orbit
- We can wait for many orbits to see the effect accumulate
- Let's calculate the perihelon advance per century, using Mercury's orbital period of 88 days:

$$\Delta\phi = 8 \times 10^{-8} \times \frac{360\mathrm{deg}}{88\textit{days}} \times \frac{36500\textit{days}}{\textit{century}} = 42.9\frac{\mathrm{arcsec}}{\textit{century}}\,. \qquad (28.12)$$

- Still a tiny number!

Mercury Perihelon Advance

- To make matters worse, there are other (non-GR) effects
- These give

$$\Delta\phi_{non-GR} = 532 \frac{\text{arcsec}}{century} \,. \tag{28.13}$$

- In total, the theory prediction (including GR) is

$$\Delta\phi_{\text{theory}} = 574.9 \frac{\text{arcsec}}{century} \,. \tag{28.14}$$

- Fortunately, observers had started to determine $\Delta\phi$ much earlier, e.g. Le Verrier (1846)
- By 1915, observed value was

$$\Delta\phi_{\text{observation}} = 575 \frac{\text{arcsec}}{century} \,. \tag{28.15}$$

A success for GR

- The theory prediction $\Delta\phi_{\rm theory}$ and observation $\Delta\phi_{\rm observation}$ match up beautifully
- Without the GR effect, there would be a clear disagreement
- The fact of the agreement served as one of the main pillars why GR became the accepted theory of gravity
- GR replaces Newton's law of gravity whenever aiming for precision; Newton's law of gravity is still used as a (very good!) approximation for many applications