Relativistic Time Delay II

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Review

- Recall that in lecture 5, we talked about time dilation in special relativity
- We found that moving clocks appear slow
- This effect is key to understanding cosmic rays
- In this lecture, we consider the *general relativistic* time delay
- This effect will be *in addition to* the special relativistic time delay

Time Delay in Special Relativity

- Let's consider a particle such as a muon with proper time increment $d\tau$
- In special relativity, where $g_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ we have

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2 = dt^2 \left(1 - \vec{v}^2\right) . \qquad (29.1)$$

- An observer sees the muon with velocity $\vec{v}$; he observes them with

$$dt = \frac{d\tau}{\sqrt{1-\vec{v}^2}} \geq d\tau . \qquad (29.2)$$

  Special relativity predicts that $d\tau = 10^{-9}s$ for the muon equals *more* than one nanosecond of dt – which is just the time delay in special relativity
- If the relative velocity vanishes $\vec{v} = 0$, $dt = d\tau$ and there is no time delay in special relativity

Time Delay in General Relativity

- Now re-consider the same example in GR using the Schwarzschild metric instead of the Minkowski metric
- For simplicity, let's have the muon travel along the radial direction only such that $d\phi = d\theta = 0$
- We have

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = A(r) dt^2 - \frac{dr^2}{A(r)} \,. \tag{29.3}$$

- If a distant observer sees muons with radial velocity $v = \frac{dr}{dt}$, the time-increment observed is

$$dt = \frac{d\tau}{\sqrt{A(r)\left(1 - \frac{v^2}{A^2}\right)}} \tag{29.4}$$

Time Delay in General Relativity

- Since $A(r) = 1 - \frac{r_s}{r}$ is smaller than one, moving clocks in general relativity *appear even slower than in special relativity*
- The GR effect of time-delay exists even if the muons are not moving at all
- Specifically, for $v = 0$, we have

$$dt = \frac{d\tau}{\sqrt{1 - \frac{r_s}{r}}} \tag{29.5}$$

- Therefore, a distant observer at $r \gg 1$ finds that a clock on earth runs slow

Time Delay for Photons

- The GR time delay exists not only for massive particles, but also for light
- For photons moving radially in a Schwarzschild geometry, we have

$$0 = -g_{\mu\nu}dx^\mu dx^\nu = A(r)dt^2 - \frac{dr^2}{A(r)} \tag{29.6}$$

- This implies

$$\frac{dt}{dr} = \frac{1}{A(r)} = \frac{1}{1-\frac{r_s}{r}}\,. \tag{29.7}$$

- A photon traveling radially outward from r to $r+\Delta r$ therefore needs a time

$$\Delta t = \int_r^{r+\Delta r} \frac{dt}{dr}dr = \int_r^{r+\Delta r} \frac{dr}{1-\frac{r_s}{r}} = \Delta r + r_s \ln\frac{r-r_s+\Delta r}{r-r_s} \tag{29.8}$$

Time Delay for Photons

- For light, the time to travel a distance Δr in Minkowski space would be $\Delta t = \Delta r$
- In GR, we find that light takes *longer* to climb out of a gravitational well