Black Hole Thermodynamics

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Review

- In the preceding lectures, we discussed Black Holes
- We calculated the Hawking temperature of a black hole in lecture 32
- In this lecture, we extend the dictionary between black holes and statistical mechanics, by considering black hole thermodynamics

Hawking Radiation

- Recall from lecture 32 that a black hole of mass M has a Hawking temperature of

$$T = \frac{1}{8\pi GM} \tag{33.1}$$

- In ordinary thermodynamics, there are relations between thermodynamic quantities, such as energy, entropy, pressure, etc.
- What do these look like for black holes?

Basic Thermodynamic Relations

Recall some basic thermodynamic relations from statistical mechanics:

- In thermodynamics, a system is characterized by its temperature T and volume V
- Thermodynamic quantities such as the energy E, the pressure P and the entropy S are related to each other
- For instance, the entropy S can be found from the pressure P as

$$S = V\frac{dP}{dT}\,, \tag{33.2}$$

  where the derivative is taken at fixed volume
- Another thermodynamic relation is

$$E = TS - PV \tag{33.3}$$

- Finally, using the relations above, one can show

$$dE = TdS\,. \tag{33.4}$$

Black Hole Energy

- Now let's apply these thermodynamic relations to black holes
- First recall that in GR, mass and energy are the same
- As a consequence, the energy of a black hole is given by

$$E = M \tag{33.5}$$

Black Hole Entropy

- Using the basic thermodynamic relations, we have for a black hole

$$dE = dM = TdS \tag{33.6}$$

- But we know the BH temperature from Hawking radiation (33.1)
- So we find

$$8\pi GMdM = dS \tag{33.7}$$

- Integrating this relation gives the black hole entropy

$$S = 4\pi GM^2 \,. \tag{33.8}$$

Bekenstein-Hawking

- Let's compare our finding for the black hole entropy to the area of a black hole
- The Schwarzschild black hole is spherical with a horizon radius given by $r_s = 2GM$
- So the area of a black hole is given by

$$A = 4\pi r_s^2 = 4\pi(4G^2M^2) = 16\pi G^2 M^2 \tag{33.9}$$

- Up to a constant factor, the area and the entropy are the same!
- We find

$$S = \frac{A}{4G}\,. \tag{33.10}$$

- This is the **Bekenstein-Hawking formula** for the entropy of a BH

Area Theorems

- We found that the entropy of a black hole is proportional to its area
- The second law of classical thermodynamics says that the entropy always has to increase,

$$dS \geq 0\,. \tag{33.11}$$

- If we take the relation to thermodynamics to be more than a mathematical curiosity, this implies for black holes

$$dA \geq 0\,. \tag{33.12}$$

- This is the **area theorem** of black holes
- It says that the area of a black hole always has to **increase**

Area Theorems and Beyond

- A consequence of the area theorem is that in BH-BH collisions, the **total area** of the final BH has to be bigger than the sum of the initial areas:

$$A_{\rm final} \geq A_{BH\ 1} + A_{BH\ 2}\,. \tag{33.13}$$

- This serves as an important check (and estimate!) for the amount of gravitational waves that can escape in such a process
- The only exception to the area theorem is Hawking radiation, which is a **quantum process** not captured by classical thermodynamics
- How an evaporating BH evades the second law of thermodynamics is known as the **information puzzle** and a subject of current research