Cosmology

paul.romatschke@colorado.edu

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Review

- We have the GR field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{34.1}$$

- So far, we have discussed *static* and *spherically symmetric* solutions for empty space ($T_{\mu\nu} = 0$) and $\Lambda = 0$
- Let's now consider relaxing the assumptions
- In particular, we now allow *time-dependent* solutions
- For simplicity, we will keep assuming spherical symmetry

Aside: maximally symmetric spaces

- For the moment, consider just 3d space, in spherical coordinates
- 3d Euclidean space in spherical coordinates:

$$ds^2 = dr^2 + r^2 d\Omega^2 \tag{34.2}$$

- Plug metric into

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 0\,. \tag{34.3}$$

- Find: 3d Euclidean space is a solution for $\Lambda = 0$
- How about solutions with $\Lambda \neq 0$?

Aside: maximally symmetric spaces

- Let's generalize our metric from 3d Euclidean:

$$ds^2 = B(r)dr^2 + r^2 d\Omega^2 \tag{34.4}$$

- Plug metric into

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 0\,. \tag{34.5}$$

- Find: Solutions for

$$B(r) = \frac{1}{1 - \Lambda r^2}\,, \quad (R = 6\Lambda) \tag{34.6}$$

Aside: maximally symmetric spaces

- For the solution

$$ds^2 = B(r)dr^2 + r^2 d\Omega^2 \tag{34.7}$$

rescale $r = \frac{\hat{r}}{\sqrt{\Lambda}}$

- One gets

$$ds^2 = \frac{1}{\Lambda} d\hat{s}^2\,, \quad d\hat{s}^2 = \frac{d\hat{r}^2}{1-\kappa\hat{r}^2} + \hat{r}^2 d\Omega^2\,, \tag{34.8}$$

where κ takes the values $\kappa = -1, 0, +1$

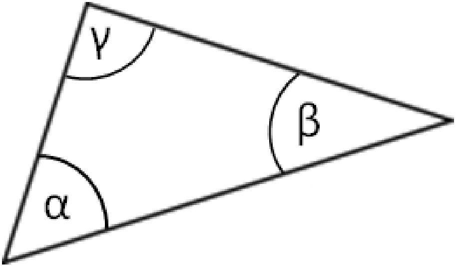
- These values correspond to three different scenarios

Aside: maximally symmetric spaces

The three different scenarios are:

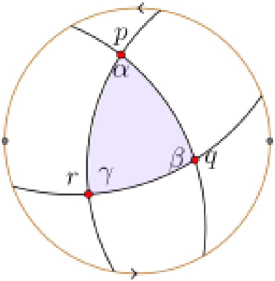
- $\kappa = 0$: Euclidean space; $\Lambda = 0$; $R = 0$: **flat space**
- $\kappa = 1$: deSitter space: $\Lambda > 0$; $R > 0$: **elliptic space**
- $\kappa = -1$: Anti-deSitter (AdS) space: $\Lambda < 0$; $R < 0$: **hyperbolic space**

Aside: Euclidean Space $\kappa = 0$



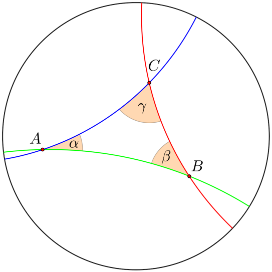
Sum of angles $\alpha + \beta + \gamma = 180^\circ$

Aside: Elliptic Space $\kappa = 1$



Sum of angles $\alpha + \beta + \gamma > 180^\circ$

Aside: Hyperbolic Space $\kappa = -1$



Sum of angles $\alpha + \beta + \gamma < 180^\circ$

Cosmology

- Let's look for solutions to GR field equations 34.1
- Simplifying assumptions:
  - **Isotropy**: (spherical symmetry)
  - **Homogeneity**: (no dependence on r)
- Minkowski (Euclidean space):

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \,. \tag{34.9}$$

- Generalize to non-Euclidean spaces:

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2 \right) \,. \tag{34.10}$$

- For $a = \mathrm{const}$, this is static

FLRW

- We want to allow dynamics
- So we allow a to be a function of time
- For this reason, we make the following ansatz:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) . \tag{34.11}$$

where $\kappa = -1, 0, 1$

- This is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric