The Friedmann Equations

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Review

- We have the GR field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{35.1}$$

- We want to find solutions to (35.1) that are dynamic, but homogeneous and isotropic
- In lecture 34, we found that such situations are captured by the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \,, \tag{35.2}$$

where $\kappa = -1, 0, 1$

Matter Content

- Besides the geometry (metric), we also need an input for the matter content
- We need to specify $T_{\mu\nu}$
- Since we are interested in *long* time-scales, we can assume matter to be in thermal equilibrium
- In addition, we will assume local gradients to be *small*

$$\partial u \ll 1\,, \quad R \ll 1\,. \tag{35.3}$$

- These assumptions lead to the *perfect-fluid form*

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}\,, \tag{35.4}$$

cf the derivation in lecture 15

Equations of Motion – Matter

- The energy-momentum tensor is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{35.5}$$

- If $T^{\mu\nu}$ is of the perfect fluid form (35.4) this entails the equations of motion

$$u^\mu \partial_\mu \epsilon + (\epsilon + p) \nabla_\mu u^\mu = 0\,. \tag{35.6}$$

- For an isotropic and homogeneous system, nothing can move, so we have

$$u^\mu = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}\,. \tag{35.7}$$

- Using the Christoffels from the FLRW metric, this implies

$$\partial_t \epsilon + (\epsilon + p) \frac{3a'(t)}{a(t)} = 0\,. \tag{35.8}$$

Equations of Motion – Geometry

- In addition to the matter EoMs, we need the GR equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{35.9}$$

- It is customary in cosmology to put the CC term **inside** $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \hat{T}_{\mu\nu} \,, \tag{35.10}$$

where

$$\hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + \left(p - \frac{\Lambda}{8\pi G}\right) g_{\mu\nu} \tag{35.11}$$

Equations of Motion – Geometry

- Einstein equations, 00 component:

$$R_0^0 - \frac{1}{2} g_0^0 R + \Lambda g_0^0 = 8\pi G T_0^0 \tag{35.12}$$

- Plugging in FLRW metric (35.2) gives

$$-3 \frac{\kappa + a'^2(t)}{a^2(t)} + \Lambda = -8\pi G \epsilon \,. \tag{35.13}$$

- These are the Friedmann Equations of Cosmology

Equations of Motion for the Universe

- We found two equations of motion for the matter part (perfect fluid) and the geometry part (Einstein Equations):

$$\partial_t \epsilon + (\epsilon + p)\frac{3a'(t)}{a(t)} = 0\,, \quad -3\frac{\kappa + a'^2(t)}{a^2(t)} + \Lambda = -8\pi G\epsilon\,. \qquad (35.14)$$

- We can introduce the shorthand notation

$$\frac{a'(t)}{a(t)} \equiv H(t)\,, \qquad (35.15)$$

so that (35.14) becomes

$$\partial_t \epsilon + 3(\epsilon + p)H = 0\,, \quad 3\left(H^2 + \frac{\kappa}{a^2}\right) = \Lambda + 8\pi G\epsilon\,. \qquad (35.16)$$

Equations of Motion for the Universe

- Cosmologists like to redefine energy and pressure by writing

$$\epsilon + \frac{\Lambda}{8\pi G} = \hat{\epsilon}\,, \quad p - \frac{\Lambda}{8\pi G} = \hat{p} \tag{35.17}$$

- Put differently, $\hat{\epsilon}, \hat{p}$ are the energy density and pressure **including the vacuum contributions**
- In terms of these, we have

$$\partial_t \hat{\epsilon} + 3(\hat{\epsilon} + \hat{p})H = 0\,, \quad 3\left(H^2 + \frac{\kappa}{a^2}\right) = 8\pi G \hat{\epsilon}\,. \tag{35.18}$$

- These are the basic equations of cosmology, and we will study their implications in the upcoming lectures!