Scale Factor Evolution

paul.romatschke@colorado.edu

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Review

- In lecture 35, we found the Friedmann equations for cosmology
- Using the shorthand notation $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ these are

$$\partial_t \epsilon + 3(\epsilon + p) H = 0\,, \quad 3\left(H^2 + \frac{\kappa}{a^2}\right) = 8\pi G \epsilon\,. \tag{37.1}$$

- In lecture 36, we identified $a(t)$ with the scale factor of the universe
- Let's get solutions to (37.1) for $a(t)$!

Dust

- Eqns. (37.1) are two equations for three unknowns (ϵ, p, a)
- In order to close the system, we need another equation
- For instance, we can use the **equation of state** (EoS) $p = p(\epsilon)$
- EoS depends on the matter content we describe
- Simplest case: "dust" (pressure-less):

$$p = 0\,. \tag{37.2}$$

- For dust, the matter part of (37.1) becomes

$$\partial_t \epsilon = -3\epsilon H \tag{37.3}$$

Other EoS

- We can also consider forms of matter other than dust
- A **very** simplified parametrization is

$$p = w\epsilon\,, \quad w = \mathrm{const.} \tag{37.4}$$

- For dust $w = 0$, while for pure radiation $w = \frac{1}{3}$
- Can also describe pure cosmological constant with $w = -1$
- In general, however, w is not a constant for real matter!
- In terms of (37.4), the matter part of (37.1) becomes

$$\partial_t \epsilon = -3H\epsilon(1+w)\,. \tag{37.5}$$

Solution to Matter Part

- For the simplified class of EoS (37.4), we can solve the matter part of the Friedmann equations
- We find

$$\epsilon(t) = \epsilon(t_0) \left(\frac{a(t_0)}{a(t)}\right)^{3(1+w)} = \epsilon_0 \left(\frac{a_0}{a}\right)^{3(1+w)} . \tag{37.6}$$

Critical Energy Density

- Plugging this into (37.1), the geometry equation reads

$$H^2 = \frac{8\pi G}{3}\epsilon_0 \left(\frac{a_0}{a}\right)^{3(1+w)} - \frac{\kappa}{a^2}\,. \tag{37.7}$$

- Dividing by H^2 we find

$$1 = \frac{8\pi G \epsilon(t)}{3H^2} - \frac{\kappa}{a^2 H^2}\,. \tag{37.8}$$

- Define the **critical energy density**

$$\epsilon_c \equiv \frac{3H^2}{8\pi G} \tag{37.9}$$

- In terms of ϵ_c, we have

$$1 = \frac{\epsilon}{\epsilon_c} - \frac{\kappa}{a^2 H^2} \tag{37.10}$$

Density Parameter

- In a proliferation of symbols, let's furthermore define the **density parameter**

$$\Omega \equiv \frac{\epsilon(t)}{\epsilon_c} \tag{37.11}$$

which is a time-dependent quantity

- In terms of Ω, we have

$$\frac{\kappa}{a^2 H^2} = \Omega - 1\,. \tag{37.12}$$

- Now $a^2 H^2 > 0$ so that the sign on both sides of this equation has to match

Density and Geometry

- Consider the implications of

$$\frac{\kappa}{a^2 H^2} = \Omega - 1\,. \tag{37.13}$$

- For a flat universe (Euclidean geometry) we have $\kappa = 0$
- For $\kappa = 0$, we must have

$$\Omega = 1 \tag{37.14}$$

  or $\epsilon = \epsilon_c$
- Using our solution for the energy density, this gives

$$\epsilon_0 \left(\frac{a_0}{a}\right)^{3(1+w)} = \frac{3H^2}{8\pi G} \tag{37.15}$$

- Since $H = \frac{\dot{a}}{a}$ we can solve this equation for $a(t)$

Flat Universe Scale Factor

- Let's solve this equation using a power-law ansatz:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\alpha}, \quad \alpha = \mathrm{const} \tag{37.16}$$

- We have

$$H = \frac{\dot{a}}{a} = \frac{\alpha}{t} \tag{37.17}$$

- Plugging this into (37.15) we have

$$\epsilon_0 \left(\frac{t}{t_0}\right)^{-3(1+w)\alpha} = \frac{3}{8\pi G}\frac{\alpha^2}{t^2} \tag{37.18}$$

- The power of t on both sides has to match:

$$\alpha = \frac{2}{3(1+w)} \tag{37.19}$$

The Flat Universe

- For the flat universe with $\kappa = 0$, we therefore find

$$a(t) \propto t^{\frac{2}{3(1+w)}} . \tag{37.20}$$

- For $w > -1$, this means the scale factor is always growing
- We have an **expanding flat universe**

The Flat Universe

- The evolution of the scale factor depends on the matter content
- For dust, where $w = 0$, we have

$$a(t) \propto t^{\frac{2}{3}} \tag{37.21}$$

- By contrast, for radiation, we have $w = \frac{1}{3}$, so we have

$$a(t) \propto t^{\frac{1}{2}} \tag{37.22}$$

- Of course, in a realistic setting we have both, and w changes depending on which form of matter dominates

Age of the Universe

- Let's look at the age of the universe again
- The beginning of the universe is fixed at $t = 0$ where $a(0) = 0$
- Today's time is t_0, and today's Hubble constant for the flat universe is

$$H_0 = \left.\frac{\dot{a}}{a}\right|_{t=t_0} = \frac{\alpha}{t_0} \tag{37.23}$$

- This implies for the age of the universe

$$t_0 = \frac{\alpha}{H_0} \tag{37.24}$$

- Plugging in $H_0 \simeq 70 \frac{km/s}{Mpc}$ this gives

$$t_0 \simeq 14\alpha Gyr \tag{37.25}$$

Age Problem Again

- For dust where $\alpha = \frac{2}{3}$ this gives

$$t_0 \simeq 9.3 Gyr \quad (37.26)$$

- For radiation, where $\alpha = \frac{1}{2}$, it's

$$t_0 \simeq 7 Gyr \quad (37.27)$$

- Since for a flat universe $w > 0$, all of these are too short for observed stars age

$$t_{\rm star} \simeq 15.2 \pm 3.7 Gyr \quad (37.28)$$

- The flat universe with $\kappa = 0$ **cannot be our universe!**