Cosmological Redshift

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Review

- In lecture 30, we discussed gravitational redshift in a Schwarzschild geometry
- In this lecture, we revisit this effect for an FLRW universe
- We will find that the scale factor and the redshift factor are related
- This lets cosmologists convert distances into redshift and vice versa

Energy of a Photon

- The energy of a photon is given by $E = \hbar\omega$
- In GR units and notation, we can write

$$E = -p^\mu u_\mu \,, \tag{38.1}$$

- Here p^μ is the momentum 4-vector of the photon
- and u^μ is the 4-velocity of the observer
- For a static observer, $u^\mu = \left(1, \vec{0}\right)$ so that $u_\mu = \left(g_{00}, \vec{0}\right)$
- For a FLRW metric, $g_{00} = -1$ so that for a static observer

$$-p^\mu u_\mu = -E\, g_{00} = E \tag{38.2}$$

A different look

- For a photon, the 4-momentum is given by

$$p^\mu = \frac{dx^\mu}{d\lambda}\,. \tag{38.3}$$

- So for a static observer in FLRW we have

$$E = \frac{x^0}{d\lambda} = \frac{dt}{d\lambda} \tag{38.4}$$

- We want to relate E to the scale factor $a(t)$ in FLRW
- We can do this by using the geodesic equation for photons

Geodesics for Photons

- Recall the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}=0\,. \tag{38.5}$$

- For photons traveling *radially* ($d\Omega=0$) in a FLRW metric this gives

$$\frac{d^2t}{d\lambda^2}+\frac{a\dot{a}}{1-\kappa r^2}\left(\frac{dr}{d\lambda}\right)^2=0\,. \tag{38.6}$$

- Since photons travel at the speed of light, we have

$$ds^2=0=-dt^2+\frac{a^2}{1-\kappa r^2}dr^2 \tag{38.7}$$

Geodesics for Photons

- So we find for the geodesic equation of radial photons

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0\,. \tag{38.8}$$

- Since $\dot{a} = \frac{da}{dt} = \frac{da}{d\lambda}\frac{d\lambda}{dt} = \frac{a'}{t'}$ we have

$$t'' + \frac{a'}{a}t' = 0\,. \tag{38.9}$$

- Substituting $t' = y$ we get

$$y' = -\frac{a'}{a}y\,, \tag{38.10}$$

- This has the solution $y = \frac{\mathrm{const}}{a}$ such that

$$\frac{dt}{d\lambda} = \frac{\mathrm{const}}{a(t)} \tag{38.11}$$

Cosmological Redshift

- We find that for radially moving photons in FLRW, the photon energy is given by

$$E = \frac{dt}{d\lambda} = \frac{\mathrm{const}}{a(t)} \tag{38.12}$$

- Since the photon's energy is proportional to its frequency, two observers at times t_1, t_2 will measure frequencies ω_1, ω_2 related by

$$\frac{\omega_1}{\omega_2} = \frac{a_2}{a_1}\,. \tag{38.13}$$

- The photon frequencies appear **redshifted** because of the expanding universe

Cosmological Redshift

- This is particularly useful if we take one observer to be *today*, e.g. $t_2 = t_0$
- We can introduce the **cosmological redshift factor**

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{a_0}{a(t)} - 1 \tag{38.14}$$

- In terms of z, we have

$$\frac{a(t)}{a_0} = \frac{1}{1+z} \,. \tag{38.15}$$

- If there is no redshift, $z = 0$, and $a(t) = a_0$
- Bigger z implies smaller $a(t)$

Small Redshifts

- Recalling the Taylor expansion of the scale factor from lecture 36 we have

$$\frac{a(t)}{a_0} \simeq 1 - (t_0 - t)H_0 + \ldots = \frac{1}{1+z}\,. \qquad (38.16)$$

- Since $t = t_0$ also implies $z = 0$, we can expand for small z as well and find

$$(t_0 - t)H_0 = z\,. \qquad (38.17)$$

- Put differently

$$H_0 = \frac{z}{t_0 - t} \qquad (38.18)$$

- For small redshifts, the Hubble constant and z are linearly related!