# TOV Equations

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# Review

- We have studied the Schwarzschild solution for a spherically symmetric static empty spacetime in lecture 22
- Stars do consist of matter, so Schwarzschild does not describe the structure of stars
- In this lecture, we derive the structure equations for static matter in GR

# GR for Static and Spherically Matter

- Let's start again with the Einstein Equations with $\Lambda = 0$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,, \tag{41.1}$$

- We will look for static & spherically symmetric solutions
- For spherically symmetric problems, it is useful to employ spherical coordinates $r, \theta, \phi$
- In spherical coordinates, the metric ansatz is

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 d\Omega^2\,. \tag{41.2}$$

(same as for Schwarzschild!)

# Static Matter

- Unlike for Schwarzschild, we need to choose a form of the energy-momentum tensor $T^{\mu\nu}$
- Since we are interested in static situations, it's a reasonable approximation to take the perfect fluid form for $T^{\mu\nu}$:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\left(u^\mu u^\nu + g^{\mu\nu}\right) . \tag{41.3}$$

- For a static fluid $u^\mu = \left(u^0, \vec{0}\right)$ so that

$$-1 = u_\mu u^\mu = -A(u^0)^2 . \tag{41.4}$$

- This implies $u^0 = A^{-\frac{1}{2}}$ or

$$T_{\mu\nu} = \mathrm{diag}\left(A\epsilon, Bp, r^2 p, r^2 p \sin^2\theta\right) . \tag{41.5}$$

# Einstein Equations

- From (41.1), we get three independent equations, such as the tt, rr and $\theta\theta$ components
- It's customary to change variables a bit by writing

$$A(r) = e^{2\alpha(r)}\,, \quad B(r) = \frac{1}{1 - \frac{2GM(r)}{r}}\,, \tag{41.6}$$

  note the similarity to $B(r) = \frac{1}{1-\frac{2GM}{r}}$ for Schwarzschild
- In terms of $\alpha, \beta$, the tt, rr components from (41.1) become

$$M'(r) = 4\pi r^2 \epsilon\,, \quad \alpha' = \frac{GM + 4\pi r^3 Gp}{r(r-2GM)}\,. \tag{41.7}$$

# Matter Equations

- We could use the third independent component from (41.1) to close the system
- However, it is much more economical to employ energy-momentum conservation:

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{41.8}$$

- For a static system, only the spatial components contribute, e.g. $\nu = r$
- Rewriting this as $\nabla_\mu T^\mu_r = 0$ we have

$$(\epsilon + p)\alpha' = -p'\,. \tag{41.9}$$

# TOV Equations

- Eliminating $\alpha$ from the matter and Einstein equations we get a set of two ordinary differential equations:

$$M'(r) = 4\pi r^2 \epsilon\,, \quad p'(r) = -(\epsilon + p)\frac{GM + 4\pi r^3 Gp}{r(r-2GM)}\,. \qquad (41.10)$$

- These are the equations of Tolmann, Oppenheimer and Volkov (simply known as **TOV equations**)
- They are 2 equations for three unknowns: $\epsilon, p, M$
- In order to solve them, we need to supply an equation of state $p = p(\epsilon)$ for the matter

# TOV equations – discussion

- The TOV equations describe the structure of matter in GR assuming hydrostatic equilibrium
- For different equations of state, they describe different objects: ordinary stars, white dwarfs and neutron stars
- Solving the TOV equations will also tell us when a star's mass becomes too great to be supported by its pressure – and it has to collapse into a black hole