Compact Stars

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Review

- In lecture 41, we derived the GR structure equations for a spherically symmetric star (TOV equations)
- In this lecture, we go on to solve the TOV equations for **compact stars** such as neutron stars

TOV Equations

- The TOV equations are given by

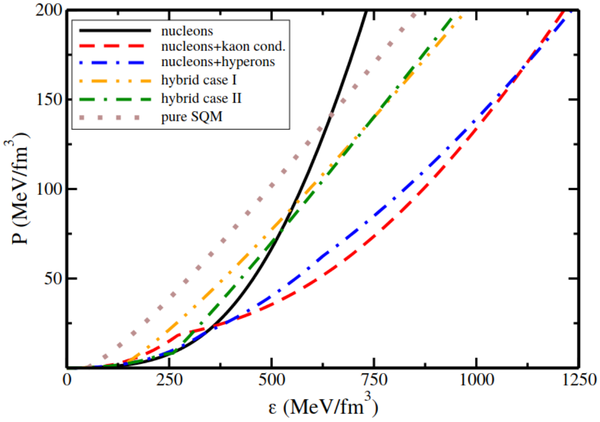
$$M'(r) = 4\pi r^2 \epsilon\,, \quad p'(r) = -(\epsilon + p)\frac{GM + 4\pi r^3 G p}{r(r-2GM)}\,. \tag{42.1}$$

- $M(r)$ is the mass within a sphere of radius r; ϵ, p are energy density and pressure
- They are 2 equations for three unknowns: ϵ, p, M
- In order to solve them, we need to supply an equation of state $p = p(\epsilon)$ for the matter

Equation of State for Nuclear Matter

- For many substances, the equation of state is readily calculated, such as for gases
- However, for others it is more tricky and requires supercomputers (e.g. high temperature QCD)
- For others still, there is **no known first principles method** to calculate $p = p(\epsilon)$ accurately
- Neutron stars fall in this last category
- However, one **can** calculate at low and high density and interpolate in the middle

Equation of State for Neutron Stars



[Kurkela, Romatschke, Vuorinen, PRD81 (2010)]

The bag model for the EoS

- Let's take a toy model for the EoS of quark stars that at least captures some of the features of real-world QCD
- In this case, we have

$$p = p(\mu) = \frac{3}{4\pi^2}\left(\frac{\mu}{3}\right)^4 - B^4\,, \quad \epsilon \equiv \mu n - P\,, \quad n \equiv \frac{\partial P}{\partial \mu}\,. \qquad (42.2)$$

- Here B is a constant (the "bag constant") that cannot be calculated
- For our toy model, we take it to be given by the fundamental QCD scale

$$B = 0.2\,\mathrm{GeV}\,. \qquad (42.3)$$

- The EoS therefore becomes

$$p = \frac{\epsilon}{3} - \frac{4}{3}B^4\,. \qquad (42.4)$$

EoS of nuclear matter

- An important feature of the nuclear matter EoS is that the pressure vanishes for a *finite* energy density $\epsilon = \epsilon_{\rm min} = 4B^4 \simeq 1 GeV/fm^3$
- We can use this to define the edge of a star by requiring that

$$p(R) = 0\,, \tag{42.5}$$

  for the surface of the star located at $r = R$
- Once we have such an equation of state we can go on to solve the TOV equations

Solving TOVs

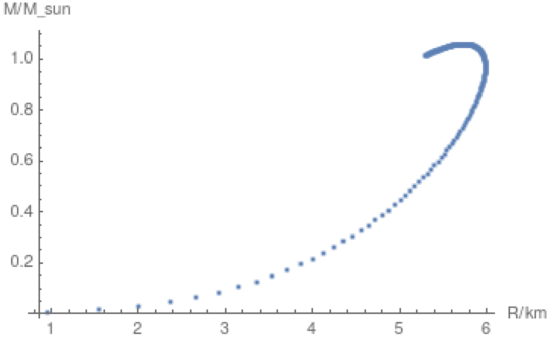
- For most situations, it is not possible to obtain analytic solutions for the TOV equations
- However, they can be readily solved *numerically*
- To do this, start with (42.1) and discretize derivatives as finite differences, e.g.

$$M'(r) = \frac{M(r+\Delta r) - M(r)}{\Delta r} = \pi (r+\Delta r)^2 \epsilon(r)\,. \tag{42.6}$$

- Starting at the star's center where $r = 0 = M$, and picking a value for $p(r=0) = p_0$ as well as a discretization Δr, we can calculate $p(\Delta r), M(\Delta r)$
- Using these as input, we can calculate the pressure and mass at $r = 2\Delta r$ and so on until we reach $p = 0$ (the edge of the star)

Quark Stars

- For each chosen value of the star's central pressure $p(r=0)=p_0$, there is a unique star radius R and a corresponding stellar mass $M(R)$
- Repeating this calculation for multiple values of p_0, one gets a family of solutions for the star's radius and mass – a mass-radius (MR) relation
- For the bag model equation of state, it looks like this:



Maximum Mass

- From the MR relation, one can read off the maximum mass of stars with a given equation of state
- For the bag model EoS with $B = 0.2$ GeV, it is $M_{\rm max} \simeq 1.05 M_\odot$
- Increasing p_0 further above the pressure that gave $M = M_{\rm max}$, the star's mass starts to *decrease* again
- These hypothetical stars beyond $M_{\rm max}$ are *unstable* against perturbations and will collapse
- Therefore stable star configurations terminate once $M_{\rm max}$ is reached

Maximum Mass – Chandrasekhar's argument

- One can give a simple argument why stars beyond $M_{\rm max}$ are unstable
- This argument does not rely on GR, so we will employ a Newtonian version
- The total energy for a star is given by its "matter" contribution $E_{\rm matter}$ minus the gravitational potential energy

$$E_{\rm tot} = E_{\rm matter} + E_{\rm grav} \tag{42.7}$$

- Assuming the star to be made up of N baryons of mass m, we have

$$M = Nm\,, \quad E_{\rm grav} = -\frac{GMm}{R} = -\frac{GNm^2}{R}\,. \tag{42.8}$$

Maximum Mass – Chandrasekhar's argument

- The matter contribution is equal to the chemical potential (Fermi energy)

$$E_{\rm matter} = \mu \propto n^{\frac{1}{3}} = \frac{N^{\frac{1}{3}}}{R} \tag{42.9}$$

- We get

$$E_{\rm tot} = \frac{N^{\frac{1}{3}}}{R} - \frac{GNm^2}{R}\,. \tag{42.10}$$

- Adding a baryon will change $E_{\rm tot}$:

$$dE_{\rm tot} = \left(\frac{1}{3}N^{-\frac{2}{3}} - Gm^2\right)\frac{dN}{R}\,. \tag{42.11}$$

Maximum Mass – Chandrasekhar's argument

- $$dE_{\rm tot} = \left(\frac{1}{3}N^{-\frac{2}{3}} - Gm^2\right)\frac{dN}{R}\,. \tag{42.12}$$
- If $N \ll 1$, then $dE_{\rm tot} > 0$; adding a baryon will increase the total energy, resulting in a stable star
- If $N \gg 1$, then $dE_{\rm tot} < 0$: adding a baryon will decrease $E_{\rm tot}$; this is a runaway process, signaling an instability
- The maximum allowed mass is given when

$$N = N_{\rm max} \simeq \left(\frac{1}{Gm^2}\right)^{\frac{3}{2}}\,. \tag{42.13}$$

Chandrasekhar limit

- From this estimate, one finds that the maximum mass of a star is given by

$$M_{\rm max} \simeq N_{\rm max} m \simeq m^{-2} \times \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}} \tag{42.14}$$

(we have restored standard units in this expression)

- This is the **Chandrasekhar limit**
- The limit depends only on the baryon mass, so it is the same for different types of stars
- For white dwarfs, one finds $M_{\rm max} \simeq 1 M_\odot$ ($R \sim 5000$ km)
- For neutron stars, one finds $M_{\rm max} \simeq 1 M_\odot$ ($R \simeq 10$ km).