Gravitational Waves I

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Review

- Recall that in lecture 20 & 21 we discussed the weak-field limit of gravity
- In those lectures, we used the weak-field limit to fix the constants in the Einstein equations
- In this lecture, we discuss the weak-field limit of the Einstein equations and will find they correspond to a wave equation for the metric field $g_{\mu\nu}$

Weak field limit of Einstein Equations

- Consider the Einstein Equations in vacuum w/o cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{44.1}$$

- Now consider the *weak field limit*

$$g_{\mu\nu}(t, \vec{x}) = \eta_{\mu\nu} + h_{\mu\nu}(t, \vec{x}) \,, \tag{44.2}$$

with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $|h_{\mu\nu}| \ll 1$

- Using symbolic manipulation software, (44.1) for $|h_{\mu\nu}| \ll 1$ becomes

$$\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \Box h_{\mu\nu} - \partial_\sigma \partial^\sigma h_{\mu\nu} - \eta_{\mu\nu} \partial_\sigma \partial_\lambda h^{\sigma\lambda} + \eta_{\mu\nu} \Box h = 0 \tag{44.3}$$

Weak field limit of Einstein Equations

- Linearizing the non-linear Einstein equations gives a second-order PDE for the metric field
- It is a wave equation for the metric field!
- However, (44.3) still looks complicated as a wave equation – this is because we have not used our diffeomorphism freedom yet

Wave equations and Gauge Fixing

- Consider electromagnetism in vacuum, for which Maxwell's equations are given by

$$0 = \partial_\mu F^{\mu\nu} = \Box A^\nu - \partial_\mu \partial^\nu A^\mu \tag{44.4}$$

(as always, note that $\Box \equiv \partial_\mu \partial^\mu$)

- We know that the the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under gauge transformations

$$A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda\,, \tag{44.5}$$

with Λ an arbitrary function of coordinates

- We can use this freedom to *fix a gauge*, e.g. demand that a condition such that $\partial_\mu A^\mu = 0$ holds. The gauge-fixed Maxwell equations then become

$$\Box A^\nu = 0\,. \tag{44.6}$$

Fixing the Gauge in Gravity

- For gravity, we don't have gauge invariance, but we have invariance under general coordinate transformations, which in many respects is very similar to gauge invariance
- So let's try to use our freedom to change coordinates to "fix the gauge" in the linearized Einstein Equations (44.3)
- In electromagnetism, we could fix a gauge such as $\partial_\mu A^\mu = \mathrm{something}$.
- Now for gravity let's also try $\partial_\mu g^{\mu\nu} = \mathrm{something}$:

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h\,, \quad h \equiv h^\mu_\mu = \eta^{\mu\nu} h_{\mu\nu}\,. \tag{44.7}$$

- In this gauge

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right)\,. \tag{44.8}$$

Gauge-Fixed Wave equation

- We can further simplify this by defining

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{44.9}$$

- In terms of the metric fluctuations $\bar{h}_{\mu\nu}$ then, the Einstein equations in vacuum become

$$\Box \bar{h}_{\mu\nu} = 0\,. \tag{44.10}$$

- We now have a wave equation for the gravitational field!
- We call the solution to this equation a gravitational wave (GW).

Speed of Propagation for Gravitational Waves

- Now that we have a wave-equation for the gravitational field, let us calculate the propagation speed for gravitational waves
- To do this, we must calculate the group velocity v_g for gravitational waves, which is defined from the dispersion relation

$$v_g = \frac{d\omega(k)}{dk}\,, \tag{44.11}$$

where ω is the frequency and k is the wave-number of the GW

- We can find the so-called *dispersion relation* $\omega(k)$ from Fourier-transforming the wave-equation:

$$\overline{h}_{\mu\nu}(t,\vec{x}) = e^{ik^\mu x_\mu}\tilde{h}_{\mu\nu}(k)\,, \quad k^\mu = \left(\omega, \vec{k}\right) \tag{44.12}$$

GW dispersion relation

- Plugging this into the wave-equation in vacuum (44.10), one finds that the wave-equation is solved by

$$\omega^2 = \mathbf{k}^2 \,. \tag{44.13}$$

- This is the dispersion relation for GW in vacuum
- From the dispersion relation, we easily calculate the group velocity for GW in vacuum:

$$v_g = \frac{d\omega}{dk} = 1 \tag{44.14}$$

  (A similar calculation shows that the phase velocity $\frac{\omega}{k} = 1$.)
- We find that in vacuum, gravitational waves propagate exactly with the speed of light

Fixing residual gauge freedom

- Picking a gauge condition does not fix the gauge completely. There is a residual gauge freedom left
- Let's take again electrodynamics as an example. Choosing temporal-axial gauge $A^0 = 0$, the Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$ for $\nu = 0$ become
$$-\partial^0 \partial_i A^i = 0\,. \tag{44.15}$$
- This means that in addition to $A^0 = 0$, we have a second condition such as
$$\partial_i A^i = 0\,. \tag{44.16}$$
- (This is to be expected: while A^μ has four components, only two of those are physical, implying that we can impose two conditions on A^μ).

Fixing residual gauge freedom – Gravity

- We have chosen the gauge condition

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \tag{44.17}$$

- By analogy with electromagnetism, we can impose a second condition on the metric fluctuation $\bar{h}_{\mu\nu}$
- We choose

$$\bar{h}^{0\nu} = 0\,, \tag{44.18}$$

which implies that only spatial components of $\bar{h}_{\mu\nu}$ are physical
- In terms of the original fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ this implies

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h\,, \quad h^{0\nu} = \frac{1}{2}\eta^{0\nu} h\,. \tag{44.19}$$

Fixing residual gauge freedom – Gravity

- Writing out the condition $h^{0\nu}=\frac{1}{2}\eta^{0\nu}h$ in components gives

$$h_{00}=-\frac{1}{2}h=-\frac{1}{2}\left(-h_{00}+h_{11}+h_{22}+h_{33}\right)\,. \tag{44.20}$$

  or $h_{00}=-\left(h_{11}+h_{22}+h_{33}\right)$. This implies $h=-2h_{00}$.
- In Fourier space, the condition $\partial_\mu h^{\mu\nu}=\frac{1}{2}\partial^\nu h$ for $\nu=0$ implies

$$\omega\tilde{h}^{00}=\frac{1}{2}\omega\tilde{h}=\,-\,\omega\tilde{h}^{00}\,, \tag{44.21}$$

  so that $h_{00}=0$.
- Since then $h=0$, in Fourier space

$$k_\mu\tilde{h}^{\mu\nu}=0\,. \tag{44.22}$$

Fixing residual gauge freedom – Gravity

- Let us now consider a gravitational wave that travels in a particular direction, say the x^3 direction. As a consequence, in Fourier space the wave-vector only has non-vanishing components $k^\mu = (\omega, 0, 0, k)$.
- The condition $k_\mu \tilde{h}^{\mu\nu} = 0$ then implies $\tilde{h}^{3\nu} = 0$
- This in turn gives

$$h = 0 = h_{11} + h_{22}\,, \tag{44.23}$$

fixing $h_{22} = -h_{11}$

Fixing residual gauge freedom – Gravity

- To summarize, for a gravitational wave traveling in the x^3 direction, we have $h_{0\nu} = 0 = h_{3\nu}$ as well as $h_{11} = -h_{22}$. Furthermore, $h_{\mu\nu} = h_{\nu\mu}$ because the metric is symmetric in the indices.
- As a consequence, after gauge fixing the metric fluctuation $h_{\mu\nu}$ only has two independent components left: h_{12} and h_{11}
- Very explicitly, we find

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{44.24}$$

- The gravitational wave has only two physical degrees of freedom (polarizations)

Gravitational Wave Polarizations