Gravitational Waves II

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Review

- In lecture 44, we discussed gravitational waves in vacuum
- This lets us discuss the propagation of gravitational waves in empty space
- In this lecture, we include matter in the Einstein equations
- This will let us discuss how gravitational waves are generated

Weak field limit of Einstein Equations

- Consider the Einstein Equations w/o cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{45.1}$$

- Now consider the *weak field limit*

$$g_{\mu\nu}(t, \vec{x}) = \eta_{\mu\nu} + h_{\mu\nu}(t, \vec{x}) \,, \tag{45.2}$$

with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $|h_{\mu\nu}| \ll 1$

- Following the same steps as in lecture 44, and using gauge fixing, (45.1) for $|h_{\mu\nu}| \ll 1$ becomes

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \,, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \,. \tag{45.3}$$

Solving the wave equation

- $$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,. \qquad (45.4)$$
- For $T_{\mu\nu} = 0$, we recover the homogeneous wave-equation from lecture 44
- For $T_{\mu\nu} \neq 0$, (45.4) inhomogeneous: it corresponds to a wave equation with a source
- We know how to solve such equations: Green's function method!

Green's function method

- Define the function $G(x-y)$ to be the solution to

$$\Box G(x-y) = \delta(x-y)\,. \tag{45.5}$$

- If $G(x)$ is known, the solution to (45.4) is given by

$$\bar{h}_{\mu\nu}(x) = -16\pi G \int d^4y G(x-y) T_{\mu\nu}(y)\,. \tag{45.6}$$

- Proof: Act with $\Box$ acting on the coordinate x on this equation:

$$\Box\bar{h}_{\mu\nu}(x) = -16\pi G \int d^4y \Box G(x-y) T_{\mu\nu}(y) \tag{45.7}$$

Green's function method

- Define the function $G(x-y)$ to be the solution to

$$\Box G(x-y)=\delta(x-y)\,. \tag{45.8}$$

- If $G(x)$ is known, the solution to (45.4) is given by

$$\bar{h}_{\mu\nu}(x)=-16\pi G\int d^4y G(x-y)T_{\mu\nu}(y)\,. \tag{45.9}$$

- Proof: Act with $\Box$ acting on the coordinate x on this equation:

$$\Box\bar{h}_{\mu\nu}(x)=-16\pi G\int d^4y\delta(x-y)T_{\mu\nu}(y) \tag{45.10}$$

Green's function method

- Define the function $G(x-y)$ to be the solution to

$$\Box G(x-y)=\delta(x-y)\,. \tag{45.11}$$

- If $G(x)$ is known, the solution to (45.4) is given by

$$\bar{h}_{\mu\nu}(x)=-16\pi G\int d^4y G(x-y)T_{\mu\nu}(y)\,. \tag{45.12}$$

- Proof: Act with $\Box$ acting on the coordinate x on this equation:

$$\Box\bar{h}_{\mu\nu}(x)=-16\pi G T_{\mu\nu}(x) \tag{45.13}$$

QED

Green's function

- There are several different functions $G(x)$ fulfilling

$$\Box G(x-y)=\delta(x-y)\,. \tag{45.14}$$

- We take the physical one, corresponding to the retarded Green's function:

$$G(x^\mu)=-\frac{\Theta(x^0)}{4\pi|\vec{x}|}\delta\left(|\vec{x}|-x^0\right)\,. \tag{45.15}$$

- Proof: homework problem
- With this Green's function, we get

$$\overline{h}_{\mu\nu}=4G\int_y \frac{T_{\mu\nu}(y)}{|\vec{x}-\vec{y}|}\Theta(x^0-y^0)\delta\left(|\vec{x}-\vec{y}|-x^0+y^0\right)\,. \tag{45.16}$$

Solution to Wave equation

- We can use the delta-function to do the y^0 integration:

$$\overline{h}_{\mu\nu}(x^0,\vec{x}) = 4G\int d^3y \frac{T_{\mu\nu}(x^0-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|} \qquad (45.17)$$

- This is the general (physical) solution for arbitrary source $T_{\mu\nu}$
- We can simplify the result for *specific* sources, such as a *localized* source

- First let's rewrite our solution for $\bar{h}_{\mu\nu}$ in frequency space:

$$\bar{h}_{\mu\nu}(x^0, \vec{x}) = \int \frac{d\omega}{2\pi} e^{-i\omega x^0} \tilde{h}_{\mu\nu}(\omega, \vec{x}) \,. \qquad (45.18)$$

- We get

$$\begin{aligned}
\tilde{h}_{\mu\nu}(\omega, \vec{x}) &= 4G \int d^3y \int dx^0 \frac{e^{i\omega x^0}}{|\vec{x}-\vec{y}|} T_{\mu\nu}\left(x^0 - |\vec{x}-\vec{y}|, \vec{y}\right) \,, \\
&= 4G \int d^3y \int dx^0 \frac{e^{i\omega x^0}}{|\vec{x}-\vec{y}|} \int \frac{d\omega'}{2\pi} e^{-i\omega'(x^0 - |\vec{x}-\vec{y}|)} \tilde{T}_{\mu\nu}\left(\omega', \vec{y}\right) \\
&= 4G \int d^3y \frac{\tilde{T}_{\mu\nu}(\omega, \vec{y})}{|\vec{x}-\vec{y}|} e^{i\omega|\vec{x}-\vec{y}|} \,. \qquad (45.19)
\end{aligned}$$

Localized Source

- If the source is localized (e.g. $T_{\mu\nu}(\omega, \vec{y}) \propto \delta(y)$) and far away, we may approximate the solution for $\tilde{h}_{\mu\nu}$
- To leading order in a localized source, we take

$$|\vec{x} - \vec{y}| \simeq |\vec{x}| = r\,. \tag{45.20}$$

- For this approximation, we get

$$\tilde{h}_{\mu\nu}(\omega, r) \simeq 4G\frac{e^{i\omega r}}{r}\int d^3y\,\tilde{T}_{\mu\nu}(\omega, \vec{y})\,. \tag{45.21}$$

- It's a spherical wave!

Localized Source – Gauge fixed

- Next recall that we have a gauge condition on the metric fluctuation:

$$\partial_\mu \bar{h}^{\mu\nu} = 0\,, \tag{45.22}$$

- In frequency space, this implies

$$i\omega \tilde{h}^{0\nu} + \partial_i \tilde{h}^{i\nu} = 0\,, \tag{45.23}$$

  which means that only spatial components $\tilde{h}^{ij}$ are independent!
- In addition to the gauge condition, recall that the energy-momentum tensor is conserved

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{45.24}$$

- In frequency space, this implies

$$i\omega \tilde{T}^{0\nu} + \partial_i \tilde{T}^{i\nu} = 0\,. \tag{45.25}$$

Energy-Momentum Tensor

- We can use this to express

$$\partial_j \tilde{T}^{0j} = -i\omega \tilde{T}^{00}\,, \quad \partial_j \tilde{T}^{ij} = -i\omega \tilde{T}^{i0}\,, \tag{45.26}$$

- Next, rewrite

$$\begin{aligned} \int d^3y\, \tilde{T}_{ij} &= \int d^3y \left[\partial_k (y_i \tilde{T}_{ij}) - y_i \partial_k \tilde{T}_{kj}\right]\,, \\ &= -\int d^3y y_i \partial_k \tilde{T}_{kj}\,, \\ &= i\omega \int d^3y y_i \tilde{T}^{0j}\,. \end{aligned} \tag{45.27}$$

Localized Source – Gauge fixed

- Collecting terms, so far we have

$$\tilde{h}^{ij}(\omega, r) = 4G\frac{e^{i\omega r}}{r} i\omega \int d^3y y^i \tilde{T}^{0j} \quad (45.28)$$

- Since $\tilde{h}^{ij} = \tilde{h}^{ji}$ we can symmetrize

$$y^i \tilde{T}^{0j} = \frac{1}{2}\left(y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i}\right) . \quad (45.29)$$

- Using the same trick as in (45.27), we have

$$\begin{aligned} \int d^3y y^i \tilde{T}^{0j} &= -\frac{1}{2}\int d^3y y^i y^j \partial_k \tilde{T}^{0k} , \\ &= \frac{i\omega}{2}\int d^3y y^i y^j \tilde{T}^{00} = \frac{i\omega}{2}\tilde{Q}^{ij} . \end{aligned} \quad (45.30)$$

Localized Source and Quadrupole Moment

- Plugging this into (45.28), we have the compact formula

$$\tilde{h}_{ij}(\omega, r) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{Q}_{ij}(\omega) \tag{45.31}$$

- Here $\tilde{Q}_{ij}$ is the *quadrupole moment* of the energy density

$$\tilde{Q}_{ij}(\omega) = \int d^3y y_i y_j \tilde{T}_{00}(\omega, \vec{y}) \,. \tag{45.32}$$

- We can convert (45.31) from frequency to the time domain, finding

$$\bar{h}_{ij}(t, r) = \frac{2G}{r} \left.\frac{d^2 Q_{ij}(t')}{dt'^2}\right|_{t'=t-r} \tag{45.33}$$