Quantum Mechanics as a Path Integral (II)

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References

- Laine and Vuorinen, “Basics of Thermal Field Theory”, Chapter 1: https://arxiv.org/pdf/1701.01554.pdf

Partition Function

Recall from Lecture 3:

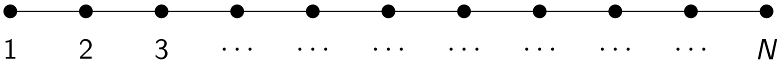
- Quantum Mechanical Partition Function

$$Z(T) = \lim_{N\to\infty} \int \left[\prod_{i=1}^{N} \frac{dx_i dp_i}{(2\pi)}\right] e^{-\epsilon \sum_{j=1}^{N}\left[\frac{p_j^2}{2m}+V(x_j)-ip_j\frac{x_{j+1}-x_j}{\epsilon}\right]}, \quad (4.1)$$

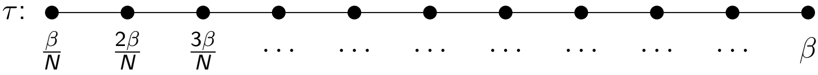
together with $x_{N+1} = x_1$.

Imaginary Time

- We can think of the N values $j = 1, 2, \ldots N$ as points on a line

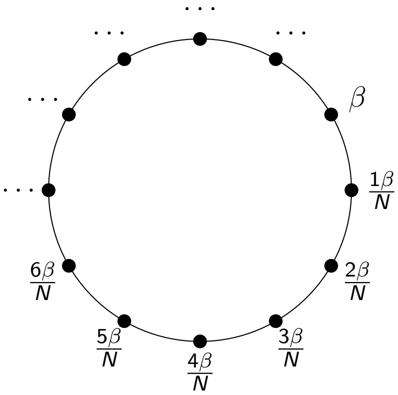


- The spacing between discrete j's is provided by $\epsilon = \frac{\beta}{N}$
- We can think of this line to represent a new coordinate τ which is discretized $\tau = j\epsilon = \frac{j\beta}{N}$, $j = 1, \ldots N$.



Imaginary Time

Recalling that $x(\beta) = x(0)$, the line closes onto itself:



This is called the "thermal circle"

Imaginary Time

- In the limit $N \to \infty$, this new coordinate τ becomes continuous: $0 \leq \tau \leq \beta$
- We will refer to the coordinate τ as "imaginary time" (for reasons that will only become clear later on)
- Using τ, we can give a contiuum meaning for the discretized expressions $\frac{x_{j+1}-x_j}{\epsilon}$, $\epsilon \sum_{j=1}^N$ appearing in Z(T) (4.1):

$$\lim_{N\to\infty} \frac{x_{j+1}-x_j}{\epsilon} = \frac{\partial x(\tau)}{\partial \tau}\,, \quad \lim_{N\to\infty} \epsilon \sum_{j=1}^{N} \to \int_0^\beta d\tau\,, \tag{4.2}$$

where $x_i \to x(\tau)$ (and p_i) have become functions of τ.

Path Integral

- Using imaginary time, the partition function (4.1) can be compactly written as

$$Z(T) = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \exp\left[-\int_0^\beta d\tau \left(\frac{p(\tau)^2}{2m} + V(x(\tau)) - ip(\tau)\frac{dx(\tau)}{d\tau}\right)\right], \tag{4.3}$$

  where $x(0) = x(\beta)$ and

$$\lim_{N\to\infty}\left[\prod_{i=1}^{N} \frac{dx_i dp_i}{(2\pi)}\right] = \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \tag{4.4}$$

- Eq. (4.3) is referred to as a contiuum "Path Integral"

Path Integral

- Eq. (4.3) contains path integration over *two* functions, position $x(\tau)$ and momentum $p(\tau)$.
- Since we consider Hamiltonians where the potential $V(x)$ is independent of momentum, it is possible to integrate out momenta

Path Integral

- Return to (4.1) and note that the momentum integrals factorize as

$$\int \frac{dp_i}{2\pi} e^{-\epsilon\left[\frac{p_i^2}{2m} - ip_i \frac{x_{i+1}-x_i}{\epsilon}\right]} = \sqrt{\frac{m}{2\pi\epsilon}} e^{-\frac{m(x_{i+1}-x_i)^2}{2\epsilon}} \,. \tag{4.5}$$

- Therefore, the path integral (4.1) becomes

$$Z(T) = \lim_{N\to\infty} \int \left[\prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi\epsilon/m)}}\right] e^{-\epsilon \sum_{j=1}^{N}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 + V(x_j)\right]} \,, \tag{4.6}$$