Quantum Mechanics as a Path Integral (III)

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References

- Laine and Vuorinen, “Basics of Thermal Field Theory”, Chapter 1:
  https://arxiv.org/pdf/1701.01554.pdf

Partition Function

Recall from Lecture 4:

- Quantum Mechanical Partition Function after integrating momenta:

$$Z(T) = \lim_{N\to\infty} \int \left[\prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi\epsilon/m)}}\right] e^{-\epsilon \sum_{j=0}^{N}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 + V(x_j)\right]}, \tag{5.1}$$

  together with $x_{N+1} = x_1$.

Partition Function

- We can again write Z(T) in continuum form

$$Z = C \int_{x(0)=x(\beta)} \mathcal{D}x \exp\left[-\int_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau))\right)\right] \tag{5.2}$$

where C is a constant factor that for fixed N is given by

$$C = \left(\frac{m}{2\pi\epsilon}\right)^{N/2} . \tag{5.3}$$

Partition Function

- The factor C is divergent for $N \to \infty$
- However, C is independent from the potential $V(x)$
- Therefore, even though C is divergent, it contains no dynamical information. We (physicists) will simply not worry about it.

Partition Function

- So we have
$$Z = C \int_{x(0)=x(\beta)} \mathcal{D}x \, e^{-S_E} \,, \tag{5.4}$$
- We call
$$S_E = \int_0^\beta d\tau L_E \,, \tag{5.5}$$
the “Euclidean” **action** and
-
$$L_E = \frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau)) \,, \tag{5.6}$$
the “Euclidean” **Lagrangian**

Partition Function

- For the moment being, these labels ("action", "Lagrangian") are rather mysterious
- Let's try to give some meaning to them by transforming our "imaginary" time variable to "real time" t:

$$\tau = it \tag{5.7}$$

We find

$$L_E \to -\frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 + V(x)\,, \tag{5.8}$$

which is nothing else but potential minus kinetic energy of a classical particle

Partition Function

- Let's turn the argument around
- Starting with the classical Lagrangian of a point particle in potential $V(x)$

$$L = \frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 - V(x) \tag{5.9}$$

we can do a transformation to imaginary time

$$t \to -i\tau \tag{5.10}$$

and get the Euclidean Lagrangian from

$$L_E = -L(t = -i\tau)\,. \tag{5.11}$$

Partition Function

- There is more:
- The action for a classical point particle is defined as

$$S = \int dt\, L\,, \tag{5.12}$$

  where L is the classical Lagrangian from before
- If we also transform the action to imaginary time, we have

$$e^{iS} \rightarrow e^{-\int d\tau L_E} = e^{-S_E}\,, \tag{5.13}$$

  with S_E the Euclidean action (5.5)
- Note: we have to put the integration limits $\tau \in [0, \beta]$ in S_E by hand

Partition Function

- To summarize, starting with the classical Lagrangian of a point particle (5.9) we get the *quantum mechanical* partition function Z(T) by writing

$$Z = C \int_{x(0)=x(\beta)} \mathcal{D}x \, e^{-S_E} \,, \tag{5.14}$$

where we get e^{-S_E} from

$$\lim_{t \to -i\tau} e^{iS} = e^{-S_E} \,, \tag{5.15}$$

and where C is a (divergent) constant