Solving The Path Integral

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References

- Laine and Vuorinen, "Basics of Thermal Field Theory", Chapter 1:
  https://arxiv.org/pdf/1701.01554.pdf

Partition Function

Recall from Lecture 5:

$$Z = C \int_{x(0)=x(\beta)} \mathcal{D}x \, e^{-S_E} \,, \tag{6.1}$$

where given a classical action S we have

$$\lim_{t \to -i\tau} e^{iS} = e^{-S_E} \,, \tag{6.2}$$

and where C is a (divergent) constant

Review

- In Lecture 2, we calculated the quantum mechanical partition function Z(T) for the harmonic oscillator using the known energy eigenvalues, finding

$$Z(T) = \frac{1}{2\sinh\left(\frac{\omega}{2T}\right)} \tag{6.3}$$

- In this Lecture, we calculate the same Z(T) by solving the path integral
- The calculation is very different, but the result has to be the same

Fourier Decomposition

- For the harmonic oscillator, the Euclidean action is

$$S_E = \int_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + \frac{m\omega^2 x^2(\tau)}{2}\right), \tag{6.4}$$

- The action is *quadratic* in the coordinate $x(\tau)$, suggesting we could use Fourier series to simplify the problem
- The periodicity of the path integral $x(0) = x(\beta)$ leads to

$$x(\tau) = T \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau} x_n \,. \tag{6.5}$$

- Here $\omega_n = 2\pi T n$ are the **Matsubara** frequencies and x_n are the Fourier coefficients
- Note: The pre-factor of temperature T is convention

Fourier Decomposition

- In addition to periodicity, we also require the coordinate $x(\tau)$ to be real, so

$$x^*(\tau) = x(\tau)\,, \quad x_n^* = x_{-n}\,. \tag{6.6}$$

- If we decompose the Fourier coefficients x_n into real and imaginary parts $x_n = a_n + ib_n$ this leads to

$$x_n^* = a_n - ib_n = a_{-n} + ib_{-n} \tag{6.7}$$

  or

$$a_n = a_{-n}\,, \quad b_n = -b_{-n}\,. \tag{6.8}$$

- Moreover, note that for the zero mode $b_0 = 0$.

Fourier Decomposition

- In the action (6.4), we have quadratic forms that give

$$\int_0^\beta d\tau\, x(\tau)y(\tau) = T^2\sum_{n,m} x_n y_m \int_0^\beta d\tau e^{i\tau(\omega_n+\omega_m)} \quad (6.9)$$

$$= T^2\sum_{n,m} x_n y_m \beta\delta_{n,-m}$$

$$= T\sum_n x_n y_{-n} = T\sum_n x_n y_n^* \quad (6.10)$$

Action – Fourier Transformed

Using the Fourier decomposition for $x(\tau)$, the action (6.4) becomes

$$\begin{aligned} S_E &= \int_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + \frac{m\omega^2 x^2(\tau)}{2}\right), \qquad (6.11)\\ &= \frac{mT}{2}\sum_n x_n \left[i\omega_n i\omega_{-n} + \omega^2\right] x_{-n}\,,\\ &= \frac{mT}{2}\sum_{n=-\infty}^{\infty}\left[-\omega_n\omega_{-n} + \omega^2\right](a_n^2 + b_n^2)\,,\\ &= \frac{mT}{2}\sum_{n=-\infty}^{\infty}\left[\omega_n^2 + \omega^2\right](a_n^2 + b_n^2)\,,\\ &= \frac{mT\omega^2}{2}a_0^2 + mT\sum_{n=1}^{\infty}\left[\omega_n^2 + \omega^2\right](a_n^2 + b_n^2)\,. \end{aligned}$$

Path Integral – Fourier Transformed

- Besides the action, we also need to transform the measure of the path integral (6.1)
- The transform is from variables $x(\tau)$ to the independent Fourier coefficients a_n, b_n
- We have

$$C\,\mathcal{D}x = C\left|\det\frac{\delta x(\tau)}{\delta x_n}\right| da_0 \left[\prod_{n=1}^{\infty} da_n db_n\right] \tag{6.12}$$

- We may regard the combination $C\left|\det\frac{\delta x(\tau)}{\delta x_n}\right|$ as another constant

$$C' = C\left|\det\frac{\delta x(\tau)}{\delta x_n}\right| \tag{6.13}$$

Path Integral – Fourier Transformed

- Putting everything together, we have

$$Z = C' \int da_0 \int \left[\prod_{n=1}^{\infty} da_n db_n\right] e^{-\frac{mT\omega^2}{2}a_0^2 - mT\sum_{n=1}^{\infty}\left[\omega_n^2+\omega^2\right](a_n^2+b_n^2)} \tag{6.14}$$

- All integrals are Gaussian:

$$\int dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\,. \tag{6.15}$$

- Therefore,

$$Z = C'\sqrt{\frac{2\pi}{mT\omega^2}}\prod_{n=1}^{\infty}\frac{\pi}{mT(\omega_n^2+\omega^2)}\,. \tag{6.16}$$

Solving the Path Integral

- The constant C' still needs to be determined
- The calculation of C' proceeds via effective field theory matching, and is assigned as a homework problem
- One finds

$$C' = \frac{T}{2\pi}\sqrt{2\pi m T}\prod_{n=1}^{\infty}\frac{mT\omega_n^2}{\pi}\,. \tag{6.17}$$

- Plugging C' into (6.16), we have

$$Z(T) = \frac{T}{\omega}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2+\omega^2} = \frac{T}{\omega}\prod_{n=1}^{\infty}\frac{1}{1+\frac{\omega^2}{(2\pi n T)^2}}\,. \tag{6.18}$$

Solving the Path Integral

- Using the identity

$$\frac{\sinh(x)}{x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{\pi^2 n^2}\right)\,, \tag{6.19}$$

in (6.18) finally gives

$$Z(T) = \frac{1}{2\sinh\left(\frac{\omega}{2T}\right)} \tag{6.20}$$

- This matches the result from Lecture 2 given in Eq. (6.3)