Quantum Mechanics and the Path Integral

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References

- Srednicki, Chapter 6

Where do we stand?

- On the one hand, we have quantum mechanics, involving operators, wave-functions and eigenvalues, e.g.

$$\hat{\mathrm{H}}\psi_n(x) = E_n\psi_n(x)\,. \tag{7.1}$$

- We can use quantum-mechanics to calculate observables, e.g.

$$Z = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} e^{-\beta E_n}\,. \tag{7.2}$$

Where do we stand?

- On the other hand, we have the path integral representation

$$Z = C \int \mathcal{D}x \, e^{-S_E} \, . \tag{7.3}$$

- All elements of the path integral are *classical*, e.g. S_E is the classical (Euclidean) action
- There are no operators, no eigenvalues, no wave-functions
- Instead there is an infinite number of integrals

Where do we stand?

- The path integral and the sum over quantized eigenvalues lead to *the same* result for Z(T)
- This is no coincidence
- In fact, the (classical) path integral and quantum mechanics are **identical** descriptions of the same physics!

Partition function vs. Transition Amplitude

- So far, we have talked about the partition function

$$Z = \mathrm{Tr}\left[e^{-\beta \hat{\mathrm{H}}}\right] = \int dx \langle x| e^{-\beta \hat{\mathrm{H}}} |x\rangle \,. \tag{7.4}$$

- What does this have to do with Quantum Field Theory?
- It is actually very similar. Consider the quantum mechanical transition amplitude from position x' at time t' to position x'' at time t'':

$$\langle x'', t'' | x, t\rangle = \langle x' | e^{-i\hat{\mathrm{H}}(t''-t')} |x\rangle \,. \tag{7.5}$$

- We can derive a path-integral for the transition amplitude exactly analogous to Z(T) by replacing $\tau \to it$

Quantum Mechanical Transition Amplitude

- Path integral for partition function

$$Z = \int dx \langle x | e^{-\beta \hat{\mathrm{H}}} | x \rangle = \mathcal{C} \int \mathcal{D}x \, e^{-\int_0^\beta L_E} \,, \quad x(\beta) = x(0) \,. \tag{7.6}$$

- Path integral for transition amplitude $\langle x'' | e^{-i \hat{\mathrm{H}}(t''-t')} | x' \rangle$:
- $\tau \to it$, implying $\int_0^\beta d\tau \to i \int_{t'}^{t''} dt$
- $x(\beta) \to x''$, $x(0) \to x'$
- This leads to

$$\langle x'' | e^{-i \hat{\mathrm{H}}(t''-t')} | x' \rangle = \mathcal{C} \int \mathcal{D}x \, e^{i \int_{t'}^{t''} L} \,, \quad x(t') = x', x(t'') = x'' \tag{7.7}$$

where L is the classical Lagrangian of a point particle

Euclidean vs. Minkowski

- The quantum mechanical transition amplitude and the quantum mechanical partition function are analogues
- The transition amplitude carries real-time information; it is formulated in Minkowski space; it's path integral is over e^{iS}; path-integral somewhat ill-defined mathematically
- The partition function carries information about thermodynamic equilibrium; it is formulated in Euclidean space; it's path integral is over e^{-S_E}; path-integral well-defined mathematically
- Because e^{-S_E} is real, $Z(T)$ can be calculated via Monte-Carlo importance sampling (computers!)

- Classical Lagrangian Mechanics: Lagrangian $L(x)$, action $S = \int dtL$
- Classical Field Theory: $x \to \phi$, Lagrangian density $\mathcal{L}(\phi)$, action $S = \int dtd^3x\mathcal{L}$
- From classical mechanics to quantum mechanics:

$$Z = \int \mathcal{D}x e^{-\int d\tau L_E(x)} \tag{7.8}$$

- By analogy, from classical field theory to quantum field theory:

$$Z = \int \mathcal{D}\phi e^{-\int d\tau d^3x \mathcal{L}_E(\phi)} \tag{7.9}$$