Classical Relativistic Field Theory

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References

- Srednicki, Chapter 8

Review of Classical Lagrangian Mechanics

- Consider a classical point particle with mass m in a one-dimensional potential $V(x)$
- The force F on the particle is given by

$$F = -\frac{dV}{dx}\,. \tag{8.1}$$

- Newton's law then gives the equations of motion for the particle

$$F = m\ddot{x} = -\frac{dV}{dx}\,, \tag{8.2}$$

where $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$

Review of Classical Lagrangian Mechanics

- We may write down the Lagrangian L for this system as

$$L(x, \dot{x}) = \text{kinetic energy} - \text{potential energy} = \frac{m}{2}\dot{x}^2 - V(x) \quad (8.3)$$

- The action for the system is given by

$$S = \int dt L \quad (8.4)$$

- Note: the equations of motion for the particle (8.2) follow from the extremum of the action S:

Review of Classical Lagrangian Mechanics

- Extremum of the action:

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \int dt \delta L(x,\dot{x}) = \int dt \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right], \\ &= \int dt \left[\frac{\partial L}{\partial x}\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \delta x \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right], \\ &= \int dt \delta x \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] + \text{boundary terms}, \quad (8.5) \end{aligned}$$

- Since any variation $\delta x(t)$ is allowed, the expression in brackets must vanish identically
- This leads to the Lagrangian equations of motion

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = -\frac{dV}{dx} - m\ddot{x}\,. \quad (8.6)$$

Relativistic Classical Field Theory

- Lagrangian mechanics is for a point particle, which is localized at a point
- Field theory is for a field that need not be localized, but can extend to all of space
- A real-world example for a field is the temperature field $T(t, \vec{x})$, which gives the temperature T at a position $\vec{x}$ and time t. The temperature is an example for a (non-relativistic) scalar field
- An example for a (non-relativistic) vector field would be the wind field $\vec{v}(t, \vec{x})$, which depends on position $\vec{x}$ and time t but has three components instead of just one
- In the following we restrict ourselves to field theories that are **relativistic** (though it is perfectly possible to consider non-relativistic field theories)

Relativistic Classical Field Theory

- A relativistic field theory has the property that its classical action S is invariant under the symmetry group of special relativity (Lorentz transformations plus translations)
- In this course, I assume you are familiar with special relativity; if not, please review chapter 2 in the textbook
- In special relativity, time and space are not independent, so writing the action as $S = \int dt L$ is not a good starting point since the volume element dt does not transform properly
- What does transform properly under the symmetries of special relativity is the volume element of space-time $dt d^3x \equiv d^4x$ such that the action for a classical field theory can be written as

$$S = \int d^4x \mathcal{L}\,, \tag{8.7}$$

where $\mathcal{L}$ is the Lagrangian **density**.

Relativistic Classical Field Theory

- In order to set up a proper relativistic field theory, we have to declare the **content** of the field theory
- Do we have scalar fields (like temperature), vector fields (like wind) or tensor fields?
- Arguably the simplest case is to start with a scalar field, which we denote by $\phi(x)$, and which corresponds to a single degree of freedom at every point in space-time x^μ
- Given the field content, we can construct a Lagrangian density $\mathcal{L}$ from $\phi(x)$ and its derivatives
- At this point, the only rule is that $\mathcal{L}$ must be a scalar under special relativity transformations

Relativistic Classical Field Theory

- If $\mathcal{L}$ contains combinations of $\phi(x)$ and $\partial_\mu\phi(x)$, allowed terms are

$$\mathcal{L} \propto \phi^2(x), \ln\phi(x), e^{-\phi(x)}, \partial_\mu\phi(x)\partial^\mu\phi(x), \partial_\mu\partial_\nu\partial_\rho\partial^\mu\partial^\nu\partial^\rho\phi \ldots \quad (8.8)$$

- On the other hand, disallowed terms would be

$$\mathcal{L} \propto \partial_\mu\phi(x), \partial_\mu\partial^\mu\phi\partial^\rho\phi \ldots \quad (8.9)$$

- To simplify the discussion, we will first limit the discussion to theories with Lagrangians of the form

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi)\,, \quad (8.10)$$

where $V(\phi)$ can be arbitrary at this point

Relativistic Classical Field Theory

- For Lagrangian densities of the form (8.10), the action is

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + V(\phi)\right] \tag{8.11}$$

- Demanding that $0 \stackrel{!}{=} \delta S$, one finds the equations of motion for the field ϕ as

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 = -\frac{dV(\phi)}{d\phi} - \partial_\mu\partial^\mu\phi\,. \tag{8.12}$$

- In the following, I sometimes denote the operator

$$\partial_\mu\partial^\mu \equiv \Box\,, \tag{8.13}$$

which is called "Quabla" operator in a wordplay on $\nabla = \partial_i\partial_i$ ("nabla")

Relativistic Classical Field Theory

- The Euclidean action for (8.11) is readily calculated using Eq. (5.11):

$$\mathcal{L}_E(\tau) = -\lim_{t\to -i\tau} \mathcal{L} = \frac{1}{2}\partial_\tau\phi\partial_\tau\phi + \frac{1}{2}\partial_i\phi\partial_i\phi + V(\phi) \tag{8.14}$$

so that

$$S_E = \int_0^\beta d\tau d^3x \left[\frac{1}{2}\partial_\tau\phi\partial_\tau\phi + \frac{1}{2}\partial_i\phi\partial_i\phi + V(\phi)\right]. \tag{8.15}$$

- Since the integrand in the Euclidean action involves a four-dimensional Euclidean quadratic form, a common notation is

$$S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right], \tag{8.16}$$

where $dx_E = d\tau d^3x$ is the Euclidean integral measure and $a = 1, 2, 3, 4$ is a four-dimensional Euclidean index