Relativistic Quantum Field Theory

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References

- Srednicki, Chapter 8

Classical Field Theory

- Recall from Lecture 8 the Euclidean action for a single scalar field

$$S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right] , \tag{9.1}$$

where $dx_E = d\tau d^3x$ is the Euclidean integral measure and $a = 1, 2, 3, 4$ is a four-dimensional Euclidean index

- Recall from Lecture 7 that the quantum theory partition function is given by

$$Z = \int \mathcal{D}\phi e^{-S_E} , \tag{9.2}$$

where periodicity on the thermal circle implies $\phi(\tau = \beta, x) = \phi(\tau = 0, x)$.

Quantum Field Theory

- Since the quantum nature is encoded in the infinite-dimensional path integral measure $\mathcal{D}\phi$, knowledge of the *classical* action provides access to the full *quantum field theory* partition function
- Unfortunately, not every classical action gives rise to a sensible quantum field theory
- Unfortunately, not every sensible quantum field theory can be solved
- Fortunately, *some* quantum field theories can be solved, so let's calculate the partition function for a free scalar quantum field!

Solving the Path Integral for Free Scalar Field Theory

- In the following, let us consider the quantum field theory for a single scalar field ϕ with potential

$$V(\phi) = \frac{m^2\phi^2}{2}\,. \tag{9.3}$$

- For this potential, the quantum field theory action (9.1) is a quadratic form in ϕ
- We can use Fourier transforms to solve the partition function just as in lecture 6!
- The only difference is that – in addition to the thermal circle parametrized by τ – we now also need to Fourier transform w.r.t. x, y, z

Solving the Path Integral for Free Scalar Field Theory

- Generalizing the setup a bit, we work in "D" spatial dimension:

$$x^i\,,\quad i=1,2,3\to x^i\,,\quad i=1,2,3,\ldots D\,. \tag{9.4}$$

- We will assume that each x^i ranges from $\left[-\frac{L}{2},\frac{L}{2}\right]$, and we will take the continuum limit $L\to\infty$ at the end
- As a consequence, we have

$$\phi(\tau,\vec{x})=\frac{T}{L^D}\sum_n\sum_{k_1,k_2,\ldots k_D}\tilde{\phi}(\omega_n,\vec{k})e^{i\omega_n\tau+i\vec{k}\cdot\vec{x}}\,, \tag{9.5}$$

where again $\omega_n=2\pi nT$ and each k_i is discretized with steps $\Delta k=\frac{2\pi}{L}$.

Solving the Path Integral for Free Scalar Field Theory

- Plugging (9.5) into the action S_E leads to

$$S_E = \frac{T}{2L^D} \sum_{\omega_n, \vec{k}} \left(\omega_n^2 + \vec{k}^2 + m^2\right) |\tilde{\phi}(\omega_n, \vec{k})|^2 \tag{9.6}$$

Since the exponent of a sum is a product, we have

$$e^{-S_E} = \prod_{\vec{k}} \exp\left[-\frac{T}{2L^D} \sum_{\omega_n} \left(\omega_n^2 + \vec{k}^2 + m^2\right) |\tilde{\phi}(\omega_n, \vec{k})|^2\right] \tag{9.7}$$

The exponent is exactly of the same form as for the quantum mechanical harmonic oscillator (6.11), with different coefficients

Solving the Path Integral for Free Scalar Field Theory

- For the harmonic oscillator (6.11), we had

$$S_E^{(HO)} = \frac{m_{(HO)}T}{2}\sum_n \left[\omega_n^2 + \omega_{(HO)}^2\right]|x_n|^2\,, \tag{9.8}$$

so that (9.6) implies the replacements

$$m_{(HO)} \to \frac{1}{L^D}\,, \quad \omega_{(HO)}^2 \to \vec{k}^2 + m^2 \equiv E_k^2\,. \tag{9.9}$$

- Using the known form of $Z(T)$ for the harmonic oscillator (6.20), we find for the quantum field theory partition function

$$Z = \prod_{\vec{k}} \frac{1}{2\sinh\left(\frac{E_k\beta}{2}\right)} = e^{-\sum_{\vec{k}}\left[\frac{E_k\beta}{2}+\ln\left(1-e^{-\beta E_k}\right)\right]}\,. \tag{9.10}$$

The Partition Function of Free Scalar QFT

- In the limit where spatial dimensions become large $L \to \infty$, the sums become integrals

$$\lim_{L\to\infty} \frac{1}{L^D} \sum_{\vec{k}} \to \int \frac{d^D \vec{k}}{(2\pi)^D} \tag{9.11}$$

- Recognizing $L^D = V$ to be the volume of D-dimensional space, this leads to the continuum free QFT partition function

$$\ln Z_{\rm free} = -\frac{V}{T} \int \frac{d^D k}{(2\pi)^D} \left(\frac{E_k}{2} + T \ln \left(1 - e^{-\beta E_k} \right) \right) . \tag{9.12}$$

- **Unlike for quantum mechanics, this result is badly divergent for any temperature!**

Making sense of QFT results

- Divergent results are a common feature of quantum field theory
- We need a procedure to see if a sensible physics answer is "hidden beneath" these divergences
- This procedure involves *regulating* the divergence ("regularization") as well as properly subtracting the divergence ("renormalization") in order to see if something sensible remains
- For the case at hand 9.12, we can understand the origin of the divergence, and properly deal with it

Making sense of QFT results

- Returning to the expression (9.10), the logarithm of the free partition function is a sum of individual harmonic oscillators with frequency E_k
- For very low temperatures $T \to 0$, the free energy $F = -T \ln Z_{\rm free}$ becomes
$$F = \sum_{\vec{k}} \frac{E_k}{2}\,, \qquad (9.13)$$
which is just a sum over the zero-point energy of all the individual harmonic oscillators
- Since in the continuum limit there is an infinite number of oscillators, and since the zero-point energy of each oscillator is non-vanishing, the resulting free energy diverges
- We understand the divergence, but it is boring: we already know about the zero-point energy in quantum mechanics

Regularizing Divergencies

- There are many ways to subtract the "boring" divergence of summing the zero-point energy of infinitely many oscillators
- A "brute-force" approach is to simply cut the sum $\sum_k$ at some maximum value of momentum $k = \Lambda$; this approach is physically well-motivated, but wreaks havoc with our cherished principle of special relativity invariance
- A much more elegant approach is to exploit our formalism for arbitrary spatial dimension D, and work in *non-integer* dimensions; this approach will be discussed in lecture 10