Regularizing QFT Divergencies

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References

- Laine and Vuorinen, "Basics of Thermal Field Theory", Chapter 2: https://arxiv.org/pdf/1701.01554.pdf

A Divergence in the free QFT partition function

- Recall from Lecture 9 that the free QFT partition function is given by

$$\ln Z_{\rm free} = -\frac{V}{T}\int \frac{d^D k}{(2\pi)^D}\left(\frac{E_k}{2} + T\ln\left(1-e^{-\beta E_k}\right)\right)\,, \qquad (10.1)$$

  where D denotes the number of space dimensions and $E_k = \sqrt{\vec{k}^2 + m^2}$.
- Instead of $Z_{\rm free}$, let us consider a physically intuitive observable, the pressure $p(T)$
- Given any partition function Z, basic thermodynamic relations define the pressure as

$$p(T) = \frac{T}{V}\ln Z\,. \qquad (10.2)$$

The pressure of free scalar quantum field theory

- From the partition function (10.1), the pressure of a single free scalar field is

$$p(T) = -\int \frac{d^D k}{(2\pi)^D} \left(\frac{E_k}{2} + T \ln \left(1 - e^{-\beta E_k} \right) \right) . \qquad (10.3)$$

- This expression for the pressure is divergent for all temperatures T
- In particular, the divergence is there also for zero temperature

$$p(0) = -\frac{1}{2} \int \frac{d^D k}{(2\pi)^D} E_k = -\frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \sqrt{\vec{k}^2 + m^2} \qquad (10.4)$$

Regulating a divergent function

- For further convenience, let us study the slightly more general function

$$\Phi(m, D, A) = \int \frac{d^D k}{(2\pi)^D} \left(\vec{k}^2 + m^2\right)^{-A} . \tag{10.5}$$

- In terms of this function, the zero-temperature free scalar field theory pressure in 3 space dimensions is given by

$$p(0) = -\frac{1}{2}\Phi\left(m, 3, -\frac{1}{2}\right) . \tag{10.6}$$

Regulating a divergent function

- Looking at the integrand of (10.5), we see that it only depends on $|\vec{k}| \equiv k$, so we can do the angular integral

$$\Phi(m, D, A) = \frac{\Omega_D}{(2\pi)^D} \int_0^\infty dk k^{D-1} \left(k^2 + m^2\right)^{-A} , \qquad (10.7)$$

  where $\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}$ is the "solid angle" in D spatial dimensions, e.g. $\Omega_3 = 4\pi$, $\Omega_2 = 2\pi$, etc.
- The integrand of (10.7) is well-behaved for any $k \in [0, \infty)$ as long as $m \neq 0$
- Assuming $m \neq 0$ in the following, the only divergence for Φ arises for high wave-number $k \to \infty$
- Borrowing from the nomenclature of light at high wavenumber, this is called an "ultraviolet" (UV) divergence
- Later on we will also encounter divergencies for $k \to 0$, which are called "infrared" (IR) divergencies

Regulating UV divergencies

- Let us first study the degree of the UV divergence of Φ
- Since the divergence is in the UV, we can concentrate on the high momentum modes k, and neglect the mass scale in the integrand
- Then we cut the momentum integral off at the scale $\Lambda \gg 1$ and find the degree of divergence as

$$\int_0^\Lambda dk k^{D-1} \left(k^2 + m^2\right)^{-A} \propto \int^\Lambda dk k^{D-1} k^{-2A} \propto \frac{\Lambda^{D-2A}}{D-2A}, \quad (10.8)$$

- In case of the pressure with $D = 3$, $A = -\frac{1}{2}$, this gives $p(0) \propto \Lambda^4$
- The pressure (10.4) diverges with degree four as $\Lambda \to \infty$
- Actually we can evaluate $p(0)$ exactly using cut-off regularization

$$p(0) = -\frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{2}\ln\left(\frac{2\Lambda}{m}\right)\right] \qquad (10.9)$$

Dimensional Regularization

- Now let us try a different regularization scheme
- Instead of introducing a UV cut-off Λ, we can realize that the integral for Φ can be solved exactly:

$$\begin{aligned}\Phi(m,D,A) &= \frac{\Omega_D}{(2\pi)^D}\frac{\Gamma\left(A-\frac{D}{2}\right)\Gamma\left(1+\frac{D}{2}\right)}{D\Gamma(A)}(m^2)^{-A+\frac{D}{2}} \\ &= \frac{1}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(A-\frac{D}{2}\right)}{\Gamma(A)}(m^2)^{-A+\frac{D}{2}}\,,\end{aligned} \tag{10.10}$$

- The divergence for $A=-\frac{1}{2}$ and $D=3$ is apparent from the divergence of the Γ function with negative argument, $\Gamma(-2)=\infty$
- But what if we work in "almost" three dimensions, e.g.

$$D=3-2\varepsilon\,, \tag{10.11}$$

with $\lim \varepsilon\to 0$?

Dimensional Regularization

- For $D = 3 - 2\varepsilon$, we have

$$\Phi(m, 3-2\epsilon, -\frac{1}{2}) = \frac{1}{(4\pi)^{\frac{3}{2}-\varepsilon}} \frac{\Gamma\left(-2+\varepsilon\right)}{\Gamma(-\frac{1}{2})} (m^2)^{2-\varepsilon}\,, \qquad (10.12)$$

which is finite for any $\varepsilon \in (0,1)$.

- We can use the property $x\Gamma(x) = \Gamma(1+x)$ of the Γ-function to write

$$\Gamma(-2+\varepsilon) = \frac{\Gamma(1+\varepsilon)}{(-2+\varepsilon)(-1+\varepsilon)\varepsilon} \qquad (10.13)$$

Dimensional Regularization

Then, we can expand all components of Φ in a power series for small ε, e.g.

$$
\begin{aligned}
(4\pi)^{-\frac{3}{2}+\varepsilon} &= \frac{2\sqrt{\pi}}{(4\pi)^2}\left[1+\varepsilon\ln(4\pi)+\mathcal{O}(\varepsilon^2)\right], \qquad (10.14)\\
(m^2)^{2-\varepsilon} &= m^4(m^2)^{-\varepsilon} = m^4\left[1-\varepsilon\ln(m^2)+\mathcal{O}(\varepsilon^2)\right],\\
\Gamma(-2+\varepsilon) &= \frac{\Gamma(1+\varepsilon)}{(-2+\varepsilon)(-1+\varepsilon)\varepsilon} = \frac{1}{2\varepsilon}\left[1+\varepsilon\left(\frac{3}{2}-\gamma_E\right)+\mathcal{O}(\varepsilon^2)\right],
\end{aligned}
$$

where $\gamma_E = 0.577215664901\ldots$ is Euler's constant.

Dimensional Regularization

- Putting everything together, we obtain for the zero-temperature pressure

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} - \ln(m^2) + \ln(4\pi) - \gamma_E + \frac{3}{2} + \mathcal{O}(\varepsilon)\right] \quad (10.15)$$

- The pressure diverges as $\varepsilon \to 0$, as with cut-off regularization
- There's only one weird thing: there is a mass squared under the logarithm; we shouldn't have a dimensionful quantity in the log!
- It's coming from the expansion of $(m^2)^{2-\varepsilon}$ in (10.14)
- And *this* is coming from the integral (10.5) having non-integer mass dimension for $D = 3 - 2\varepsilon$

Dimensional Regularization

- To avoid a dimensionful quantity under the logarithm, we re-define the function Φ from (10.5) in $D = 3 - 2\varepsilon$ dimensions as

$$\Phi(m, 3-2\varepsilon, A) = \mu^{2\varepsilon} \int \frac{d^{3-2\varepsilon}k}{(2\pi)^{3-2\varepsilon}} \left(\vec{k}^2 + m^2\right)^{-A} . \qquad (10.16)$$

- The parameter μ is arbitrary, and is taken to have dimensions of energy (mass)
- This ensures that the pressure has mass dimension four even if $\varepsilon \neq 0$
- We call μ the renormalization scale parameter

Dimensional Regularization

- With the redefinition (10.16) for non-integer dimensions, the dimensionally-regulated zero-temperature pressure then reads

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + \ln(4\pi) - \gamma_E + \frac{3}{2} + \mathcal{O}(\varepsilon)\right] \quad (10.17)$$

- It is customary to further simplify this expression by introducing instead of μ the so-called "$\overline{\mathrm{MS}}$ scheme" scale parameter

$$\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E} \quad (10.18)$$

- In $\overline{\mathrm{MS}}$ then

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2} + \mathcal{O}(\varepsilon)\right] \quad (10.19)$$