Renormalization

paul.romatschke@colorado.edu

PHYS 7270, Fall 2020

Regulated QFT Observables

- Recall from Lecture 10 that we can regulate the divergencies in QFT with different methods
- Using cut-off regularization, we found for the pressure of a single free scalar field (10.9):

$$p(0) = -\frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{4}\ln\left(\frac{4\Lambda^2}{m^2}\right)\right]. \qquad (11.1)$$

- Using dimensional regularization in $\overline{\mathrm{MS}}$, we found (10.19):

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2} + \mathcal{O}(\varepsilon)\right]. \qquad (11.2)$$

Regulated QFT Observables

- Because we used different regularization schemes, the expressions (11.1), (11.2) look different
- For $\Lambda \to \infty$, (11.1) becomes large and *negative*
- For $\varepsilon \to 0$, (11.2) becomes large and *positive*
- In fact, the only thing that is identical between (11.1), (11.2) is the coefficient of the logarithm, $\frac{m^4}{64\pi^2}$
- How can this be? Shouldn't an observable such as the pressure be independent from the regularization scheme?

Renormalization – poor man's version

- Recall that all we've been talking about so far is the divergence in the pressure at zero temperature
- This is the pressure of the vacuum (a.k.a. "cosmological constant")
- The actual pressure is temperature-dependent, and given by (10.3):

$$p(T) = p(0) - T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\beta E_k}\right) . \qquad (11.3)$$

- But except for $p(0)$, the above integral is convergent!
- If we are willing to just consider the *difference* of $p(T) - p(0)$, then we have a perfectly fine quantum field theory result!

Renormalization – poor man's version

- Instead of normalizing the pressure absolutely, we could *renormalize* this observable by demanding that the pressure of the vacuum is zero
- We could call this quantity the "renormalized" pressure

$$p^{\rm ren}(T) \equiv p(T) - p(0) = -T \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\beta E_k}\right) \equiv -J_B(T, m)\,. \tag{11.4}$$

- The renormalized pressure is finite for any temperature. In the case of vanishing mass (m=0), we can evaluate it analytically as a homework problem:

$$\lim_{m\to 0} p^{\rm ren}(T) = -J_B(T, 0) = \frac{\pi^2 T^4}{90}\,. \tag{11.5}$$

Renormalization – poor man's version

- In this "poor-man's version" of renormalization, we have

$$p^{\rm ren}(T=0)=0\,. \tag{11.6}$$

- While this is not a bad choice, it's a choice nonetheless
- For instance, we could have chosen the renormalized pressure to vanish at a temperature *different from* $T=0$
- In fact, any renormalization condition of the form

$$p^{\rm ren}(T=\mu)=0\,, \tag{11.7}$$

works, where we call μ the renormalization scale

Renormalization – general remarks

- For renormalization of the pressure, we need a condition at an arbitrary scale μ
- Another word for "condition" is scheme, so we need to pick a "renormalization scheme"
- We also need a scale at which we evaluate the condition, which is called the "renormalization scale"
- Different choices of renormalization scale/scheme lead to different renormalized observables; for instance, consider (11.7) once for $\mu_1 = 0$ and once for $\mu_2 = 1$ eV. The corresponding renormalized pressure functions $p_1(T), p_2(T)$ will differ from each other by a constant

Towards a More Formal View of Renormalization

- So far we have considered a "poor-man's" version of renormalization where we simply discard the divergent part of our QFT result for the pressure
- However, we can proceed in a more systematic fashion
- Recall that in lecture 8, we said that a large number of terms was allowed for the classical Lagrangian $\mathcal{L}$, cf. Eq. (8.8), but we chose to restrict to the form (8.10) for simplicity
- Let's now revisit this choice, and consider the slightly more general Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi) - K\,, \tag{11.8}$$

where K is an overall constant (we could equally well take K to be part of $V(\phi)$)

Towards a More Formal View of Renormalization

- Using again a potential $V(\phi) = \frac{m^2\phi^2}{2}$ as in lecture 9, we have the Euclidean action

$$S_E = \int d^4 x_E \left[\frac{1}{2} \partial_a \phi \partial_a \phi + \frac{m^2}{2} \phi^2 + K \right] \tag{11.9}$$

- Following exactly the same steps as in lecture 9, we obtain a modified expression for the pressure:

$$\begin{aligned} p(T) &= -K - \int \frac{d^D k}{(2\pi)^D} \left(\frac{E_k}{2} + T \ln \left(1 - e^{-\beta E_k} \right) \right) \\ &= -K + p(0) - J_B(T, m) \,. \end{aligned} \tag{11.10}$$

Towards a More Formal View of Renormalization

- Let us first regularize the divergence in $p(0)$ using the momentum cutoff, cf. (11.1):

$$p(T) = -K - J_B(T,m) - \frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{4}\ln\left(\frac{4\Lambda^2}{m^2}\right)\right], \tag{11.11}$$

  and recall that $J_B(T,m)$ is finite for all T
- Since the constant K is arbitrary, we can choose it as we want
- In particular, we can choose K to be *divergent* as $\Lambda \to \infty$
- In the cut-off renormalization scheme and scale parameter $\mu_{\rm cut}$, let us therefore choose

$$K = -\frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{4}\ln\left(\frac{4\Lambda^2}{\mu_{\rm cut}^2}\right)\right] \tag{11.12}$$

Towards a More Formal View of Renormalization

- Now let's redo the calculation in dimensional regularization, cf. (11.2):

$$p(T) = -K - J_B(T,m) + \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right) + \frac{3}{2}\right], \qquad (11.13)$$

so that in $\overline{\mathrm{MS}}$ we choose

$$K = \frac{m^4}{64\pi^2}\frac{1}{\varepsilon} \qquad (11.14)$$

(There is a reason why it is called "minimal-subtraction" scheme)

Towards a More Formal View of Renormalization

- In cut-off renormalization, we therefore have

$$p^{\rm ren}(T,\mu_{\rm cut}) = -J_B(T,m) + \frac{m^4}{64\pi^2}\ln\left(\frac{\mu_{\rm cut}^2}{m^2}\right)\,, \tag{11.15}$$

  where we have made the dependence on the renormalization scale parameter $\mu_{\rm cut}$ explicit
- By contrast, in $\overline{\rm MS}$ we get

$$p^{\rm ren}(T,\bar\mu) = -J_B(T,m) + \frac{m^4}{64\pi^2}\ln\left(\frac{\bar\mu^2 e^{\frac{3}{2}}}{m^2}\right) \tag{11.16}$$

- The results for $p^{\rm ren}$ are identical if we identify

$$\mu_{\rm cut}^2 = e^{\frac{3}{2}}\bar\mu^2 \tag{11.17}$$

Renormalization

- We can use the freedom in the classical Lagrangian $\mathcal{L}$ to add additional terms called "counterterms"
- We need a regularization scheme to handle divergent integrals (e.g. Λ, ϵ)
- We can let the counterterms depend on the regularization parameter and use them to "cancel" the divergent part of the integrals
- We are left with *finite* ("renormalized") observables that in general still depend on the renormalization scale
- Renormalized quantities are different in different renormalization schemes; however, it is possible to convert between renormalization schemes through converting the renormalization scale

The Cosmological Constant

- Regardless of the renormalization scale chosen, the renormalized pressure depends is scale dependent
- For free scalar quantum field theory, this cannot be avoided as long as $m \neq 0$
- Recall that the $T = 0$ pressure is basically the cosmological constant: the most common interpretation of this “feature” is that the cosmological constant depends on scale
- While we cannot calculate it’s value, our result suggests that the cosmological constant is not constant; it changes with time!