Interacting QFT and Perturbation Theory

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Review from past lectures

- In lecture 7, we found the partition function for a quantum field theory with a single scalar field to be given by

$$Z = \int \mathcal{D}\phi e^{-S_E} \tag{12.1}$$

- Here S_E is the Euclidean action of the corresponding *classical* relativistic field theory; we limited our discussion to second-order derivative theories, e.g. those with

$$S_E = \int d^4 x_E \left[\frac{1}{2} \partial_a \phi \partial_a \phi + V(\phi) \right] \tag{12.2}$$

- In lecture 9, we calculated Z for *free* field theory, e.g.

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \tag{12.3}$$

Interacting QFT

- Now let us consider *interacting* field theories
- These are field theories where $V(\phi)$ contains terms other than ϕ^0, ϕ^1, ϕ^2
- While it's possible to consider complicated functional forms for $V(\phi)$, let us restrict our attention to power-like terms, e.g. ϕ^n with integer n for simplicity
- For a better understanding, let us classify contributions to $V(\phi)$ according to their mass dimension

Operator dimensions in QFT

Recall the form of the Euclidean action:

$$S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right] \tag{12.4}$$

- The action is dimensionless, which I'll write as $[S_E] = 0$
- Since it is equal to an integral over 4 space-time dimensions, the *integrand* must have mass dimension 4 in order for S_E to be dimensionless, so

$$[\partial_a\phi\partial_a\phi] = 4, \quad [V(\phi)] = 4\,. \tag{12.5}$$

- Derivatives have mass dimension one, $[\partial_a] = 1$, so it follows that

$$[\phi] = 1 \tag{12.6}$$

Operator dimensions in QFT

- While ϕ here still is a classical field, once we plug it into the path integral it will correspond to an operator
- When we count mass dimensions of "operators" in the classical field theory we call these "naive" mass dimension (because quantum mechanical corrections may change them in some cases)
- Nevertheless, the naive operator dimensions are a good starting point
- Since $[\phi] = 1$ and $[V(\phi)] = 4$, an obvious choice for $V(\phi)$ in 4 dimensions is

$$V(\phi) = \lambda\phi^4\,. \qquad (12.7)$$

- Note that $[\lambda] = 0$, so the coupling is dimensionless

Relevant, Irrelevant and Marginal Operators

- If the operator has mass dimensions exactly equal to the number of space-time dimensions, its coupling is dimensionless;this is the case for $V(\phi) = \lambda\phi^4$ in four dimensions
- A different nomenclature calls these special operators "marginal"
- By contrast, operators with mass dimension less than the the number of space-time dimensions are called "relevant";an example for a relevant operator would be ϕ^2
- By dimensional counting, the coupling for relevant operators must have positive mass dimension;for the case of ϕ^2 in four dimensions, the coupling is simply m^2
- Finally, operators with mass dimension exceeding the number of space-dimensions are called "irrelevant"

Poster-child for interacting QFT

- A good starting point for an interacting QFT is using marginal operators
- Don't be fooled by the name; it turns out that marginal operators are the most important operators to consider
- For the case of a single scalar field in four dimensions, in the following we take
$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4\,, \tag{12.8}$$
and try to solve the corresponding quantum field theory
- Note that for $\lambda = 0$, this exactly corresponds to the free QFT we solved in lecture 9

Poster child for interacting QFT

- With the potential fixed by (12.8), the partition function for the interacting quantum field theory becomes

$$Z = \int \mathcal{D}\phi e^{-S_0 - S_I}\,, \tag{12.9}$$

where

$$\begin{aligned} S_0 &= \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + \frac{1}{2}m^2\phi^2\right]\,, \\ S_I &= \lambda \int d^4x_E \phi^4\,. \end{aligned} \tag{12.10}$$

- We will restrict the coupling $\lambda \geq 0$, because for $\lambda < 0$ the integral over ϕ does not converge

Poster child for interacting QFT

- For the *discretized* path integral where

$$\mathcal{D}\phi = \prod_{i=1}^{N} [d\phi_i] \ , \tag{12.11}$$

  the partition function Z_N is well-defined for any finite N

- However, because the term S_I is not quadratic in the field ϕ, Fourier transforms are not going to help us solve Z
- The math to solving non-Gaussian integrals has simply not been invented!
- So instead of solving for Z, we lower our goal: let's try to solve for Z *approximately*

Perturbation Theory

- We are trying to approximate

$$Z=\int \mathcal{D}\phi e^{-S_0-S_I}\,, \tag{12.12}$$

where we can integrate $\int \mathcal{D}\phi e^{-S_0}$

- Let's try expanding the exponential:

$$\begin{aligned} Z &= \int \mathcal{D}\phi e^{-S_0}\left(1-S_I+\frac{1}{2}S_I^2-\frac{1}{3!}S_I^3+\ldots\right)\,, \\ &= Z_{\rm free}\left(1-\langle S_I\rangle+\frac{1}{2}\langle S_I^2\rangle-\frac{1}{3!}\langle S_I^3\rangle+\ldots\right)\,. \end{aligned} \tag{12.13}$$

where

$$Z_{\rm free}=\int \mathcal{D}\phi e^{-S_0}\,,\quad \langle\ldots\rangle\equiv\frac{\int \mathcal{D}\phi[\ldots]e^{-S_0}}{Z_{\rm free}}\,.$$

Perturbation Theory

- The expansion of the exponential (12.13) is a power series in the coupling λ
- For small λ, we can treat S_I as a perturbation around S_E, hence we call the series in (12.13) "perturbative series"
- It is not guaranteed that the perturbative series converges
- But in many cases we find that the series is at least asymptotic
- Assuming the series is asymptotic, the perturbative series will let us approximate the interacting QFT provided that λ is small

Perturbation Theory

- To first order in perturbation theory, the partition function is given by

$$Z \simeq Z_1 = Z_{\rm free}\left(1 - \langle S_I\rangle\right) = Z_{\rm free}\left(1 - \lambda \int_x \langle \phi^4(x)\rangle\right) \tag{12.14}$$

- We need to calculate

$$\langle \phi^4(x)\rangle \tag{12.15}$$

which we will do via Wick's theorem (see lecture 13)

Perturbation Theory

- Higher order perturbative results can be calculated in a similar fashion, e.g. at second order

$$
\begin{aligned}
Z_2 &= Z_{\rm free}\left(1-\langle S_I\rangle+\frac{1}{2}\langle S_I^2\rangle\right)\,, && (12.16)\\
&= Z_{\rm free}\left(1-\lambda\int_x\langle\phi^4(x)\rangle+\frac{\lambda^2}{2}\int_{x,y}\langle\phi^4(x)\phi^4(y)\rangle\right) && (12.17)
\end{aligned}
$$

- This will require us to calculate

$$\langle\phi^4(x)\phi^4(y)\rangle \qquad (12.18)$$

Perturbation Theory

- Once we have a perturbative result for Z, we can calculate obtain other observables in perturbation theory
- For instance, the pressure to third order in perturbation theory is given by

$$\begin{aligned} p &= \frac{1}{\beta V}\ln Z = \frac{1}{\beta V}\left[\ln Z_{\rm free} + \ln\left(1 - \langle S_I\rangle + \frac{1}{2}\langle S_I^2\rangle - \frac{1}{3!}\langle S_I^3\rangle\right)\right]\,, \\ &= p_{\rm free} + \frac{1}{\beta V}\left[-\langle S_I\rangle + \frac{1}{2}\left(\langle S_I^2\rangle - \langle S_I\rangle^2\right)\right. \\ &\qquad \left. -\frac{1}{3!}\left(\langle S_I^3\rangle - 3\langle S_I^2\rangle\langle S_I\rangle + 2\langle S_I\rangle^3\right)\right]\,. \end{aligned} \tag{12.19}$$

- In perturbation theory, all observables are power series in λ