Wick's theorem

paul.romatschke@colorado.edu

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Review

- In lecture 12, we found that we need to calculate

$$\langle\phi^4(x)\rangle, \quad \langle\phi^4(x)\phi^4(y)\rangle\,, \quad \ldots \tag{13.1}$$

where expectation values w.r.t. the free action S_0 are defined as

$$\langle\ldots\rangle \equiv \frac{\int \mathcal{D}\phi[\ldots]e^{-S_0}}{Z_{\rm free}}\,. \tag{13.2}$$

- We will calculate these using Wick's theorem in this lecture

Wick's theorem

- Wick's theorem states that expectation values w.r.t. to a Gaussian action can be decomposed as

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \sum_{\text{all combinations}} \langle\phi(x_1)\phi(x_2)\rangle\ldots\langle\phi(x_{n-1})\phi(x_n)\rangle \tag{13.3}$$

- For the example of the pressure discussed in lecture 12, this implies for instance

$$\langle\phi^4(x)\rangle = 3\left(\langle\phi^2(x)\rangle\right)^2 . \tag{13.4}$$

- Let's prove Wick's theorem now:

Gaussian integrals

- Consider integrals over a vector field v_i that are Gaussian,

$$\int d\mathbf{v} e^{-\frac{1}{2}v_i A_{ij} v_j + b_i v_i} \equiv e^{W[b]}\,, \tag{13.5}$$

  where A_{ij} is a positive-definite symmetric matrix so that the integral converges, and b_i is an arbitrary vector
- We can evaluate the Gaussian integral via variable substitution: $v_i \to v_i + A^{-1}_{ij} b_j$ such that

$$e^{W[b]} = e^{-\frac{1}{2}b_i A^{-1}_{ij} b_j} \int d\mathbf{v} e^{-\frac{1}{2}v_i A_{ij} v_j} = e^{-\frac{1}{2}b_i A^{-1}_{ij} b_j} e^{W[0]} \tag{13.6}$$

Generating Functions

- We will call $e^{W[b]}$ the "generating function"
- We can derive $e^{W[b]}$ w.r.t. the vector b_i to generate expectation values of v_i because

$$\frac{\partial}{\partial b_i} e^{W[b]} = \int d\mathbf{v} \frac{\partial}{\partial b_i} e^{-\frac{1}{2} v_i A_{ij} v_j + b_i v_i} = \int d\mathbf{v} v_i e^{-\frac{1}{2} v_i A_{ij} v_j + b_i v_i} \quad (13.7)$$

so that

$$\left. \frac{\partial}{\partial b_i} e^{W[b]} \right|_{b=0} = \int d\mathbf{v} v_i e^{-\frac{1}{2} v_i A_{ij} v_j} = \langle v_i \rangle e^{W[0]} \quad (13.8)$$

Generating Functions

- Taking derivatives w.r.t to b_i can be repeated arbitrarily so

$$\frac{\partial}{\partial b_i \partial b_k \partial b_l \ldots \partial b_n} e^{W[b]}\Big|_{b=0} = \langle v_i v_k v_l \ldots v_n \rangle e^{W[0]}\,. \qquad (13.9)$$

- But we calculated $e^{W[b]}$ earlier in (13.6); plugging it in leads to

$$\begin{aligned}\langle v_i v_k v_l \ldots v_n \rangle &= \frac{\partial}{\partial b_i \partial b_k \partial b_l \ldots \partial b_n} e^{-\frac{1}{2} b_m A^{-1}_{mq} b_q}\Big|_{b=0}\,, \qquad (13.10)\\ &= \frac{\partial}{\partial b_i \partial b_k \partial b_l \ldots \partial b_n} \Bigg(1 - \frac{1}{2} b_m A^{-1}_{mq} b_q \\ &\quad + \frac{1}{2}\left(\frac{1}{2}\right)^2 b_m A^{-1}_{mq} b_q b_r A^{-1}_{rs} b_s + \ldots\Bigg)\Bigg|_{b=0}\,.\end{aligned}$$

Wick's theorem

Having expanded the exponential of the quadratic form in b, we immediately obtain the following results for the expectation values

- $\langle 1\rangle = 1$
- $\langle v_i\rangle = 0$, $\langle v_i v_k v_l\rangle = 0, \ldots$; any odd number of operators v_i has vanishing expectation value
- $\langle v_i v_j\rangle = A^{-1}_{ij}$
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$$\begin{aligned} \langle v_i v_j v_m v_n\rangle &= A^{-1}_{ij}A^{-1}_{mn} + A^{-1}_{im}A^{-1}_{jn} + A^{-1}_{in}A^{-1}_{jm} \\ &= \langle v_i v_j\rangle\langle v_m v_n\rangle + \langle v_i v_m\rangle\langle v_j v_n\rangle + \langle v_i v_n\rangle\langle v_j v_m\rangle\,. \end{aligned} \tag{13.11}$$

- Expectation values of even multiples of v_i generalize (13.11) such that we get a sum over all possible combinations of two-point expectation values

Wick's theorem

- If we let the number of vectors v_i tend to infinity, we obtain a continuous field $v(x)$
- The results on the previous slide remain unchanged in this limit
- Renaming $v(x) \to \phi(x)$, this proves Wick's theorem (13.3)