# Propagator in Scalar Field Theory I

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## Review

- In lecture 14, we found a perturbative expression for the pressure of scalar field theory with $\phi^4$ interaction
- To first order in perturbation theory we had (14.4):

$$p_{(1)} = p_{\rm free} - \frac{3\lambda}{\beta V}\int_x \left(\langle\phi(x)\phi(x)\rangle\right)^2 \, . \tag{15.1}$$

- We still have not discussed on how to calculate the propagator $\langle\phi(x)\phi(y)\rangle$, which is the subject of this lecture

# The Free Propagator

- The (free) propagator is defined as

$$\langle \phi(x)\phi(y)\rangle = \frac{\int \mathcal{D}\phi e^{-S_0}\phi(x)\phi(y)}{Z_{\rm free}}\,, \tag{15.2}$$

where $S_0 = \int_x \left[\frac{1}{2}\partial_a\phi\partial_a\phi + \frac{1}{2}m^2\phi^2\right]$ and $Z_{\rm free} = \int \mathcal{D}\phi e^{-S_0}$

- The path integral is Gaussian, so the result will involve the inverse of the operator $\partial_a\partial_a + m^2$; it is easier to express this inverse in momentum space

## The Free Propagator

- Since $S_0$ is quadratic in the field $\phi$, we employ once again a Fourier-transform (cf. (9.5)):

$$\phi(x) = \frac{T}{V}\sum_{\omega_n,\vec{k}} e^{i\omega_n\tau + i\vec{k}\cdot\vec{x}}\tilde{\phi}(\omega_n,\vec{k}) \tag{15.3}$$

- For notational simplicity, I will introduce $K = \left(\omega_n,\vec{k}\right)$ as the Euclidean 4-momentum
- The free propagator then is

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{\beta^2 V^2}\sum_{P,K}\frac{\int\mathcal{D}\tilde{\phi}e^{-S_0}\tilde{\phi}(P)\tilde{\phi}(K)e^{iP\cdot x+iK\cdot y}}{Z_{\rm free}} \tag{15.4}$$

# The Free Propagator

- The action in Fourier-space is (cf. (9.6))

$$S_0 = \frac{1}{2\beta V} \sum_K \left(K^2 + m^2\right) |\tilde{\phi}(K)|^2 \tag{15.5}$$

- Recalling from lecture 13 that the Gaussian integral $\int d\mathbf{v} e^{-\frac{1}{2} v_i A_{ij} v_j} v_m v_n$ is vanishing whenever $m \neq n$, this implies that the propagator is only non-vanishing whenever $\tilde{\phi}(P)\tilde{\phi}(K) = |\tilde{\phi}(K)|^2$
- Since $\tilde{\phi}(P) = \tilde{\phi}^*(-P)$ because of periodic boundary conditions this implies $P + K = 0$ for a non-vanishing propagator

# The Free Propagator

- One therefore has

$$\int \mathcal{D}\tilde{\phi} e^{-\frac{1}{2\beta V}\sum_Q \left(Q^2+m^2\right)|\tilde{\phi}(Q)|^2} \tilde{\phi}(P)\tilde{\phi}(K) = Z_{\rm free} \times \frac{\beta V \delta_{P,-K}}{K^2+m^2}\,, \quad (15.6)$$

  where $\delta_{i,j}$ denotes the Kronecker-delta
- The free propagator therefore becomes

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{\beta V}\sum_K \frac{e^{iK\cdot(y-x)}}{K^2+m^2} = \frac{1}{\beta V}\sum_K \frac{e^{iK\cdot(x-y)}}{K^2+m^2}\,, \quad (15.7)$$

  where the last identity follows from $\sum_K = \sum_{-K}$

# The Free Propagator

- The propagator $\langle\phi(x)\phi(y)\rangle$ only depends on the difference $x-y$; this is a consequence of translational invariance of the operator $\Box + m^2$
- In the large volume limit $V\to\infty$, the sum over wave-numbers becomes an integral:

$$\langle\phi(x)\phi(y)\rangle = T\sum_{\omega_n}\int \frac{d^D k}{(2\pi)^D}\frac{e^{iK\cdot(x-y)}}{K^2+m^2} = G_{\rm free}(x-y)\,, \qquad (15.8)$$

  where $G_{\rm free}(X)$ is a new notation for the free propagator
- In the zero-temperature limit, we have by analogy

$$\lim_{T\to 0} G_{\rm free}(x-y) = \int \frac{d^{D+1}K}{(2\pi)^{D+1}}\frac{e^{iK\cdot(x-y)}}{K^2+m^2} \qquad (15.9)$$

# Perturbative Pressure Correction

- We now have all the tools in place to calculate the perturbative pressure (15.1)
- Eq. (15.8) leads to

$$\langle \phi(x)\phi(x)\rangle = G_{\rm free}(0) = T\sum_{\omega_n}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{K^2+m^2} \tag{15.10}$$

- There are techniques to calculate $G_{\rm free}(0)$ directly, but we will use a trick

# Perturbative Pressure Correction

- Noting that

$$\frac{1}{K^2+m^2}=\frac{\partial}{\partial m^2}\ln\left(K^2+m^2\right) \tag{15.11}$$

we have

$$G_{\rm free}(0)=\frac{\partial}{\partial m^2}T\sum_{\omega_n}\int_k \ln\left(\omega_n^2+k^2+m^2\right)\,. \tag{15.12}$$

- But from lectures 6 and 9, we had (cf. (6.18), (9.10))

$$Z_{\rm free}=\prod_{\vec{k}}\frac{T}{E_k}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2+E_k^2} \tag{15.13}$$

so that

$$\ln Z_{\rm free}=-\frac{1}{2}\sum_{\vec{k}}\sum_{n=-\infty}^{\infty}\ln(\omega_n^2+k^2+m^2)+{\rm m-indep.} \tag{15.14}$$

# Perturbative Pressure Correction

- Therefore

$$G_{\rm free}(0) = -2\frac{\partial}{\partial m^2} p_{\rm free} \tag{15.15}$$

- Using the result (11.16) for $p_{\rm free}$ that we calculated in lecture 11 in dimensional regularization, we therefore find

$$G_{\rm free}(0) = -\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar\mu^2 e^{\frac{1}{2}}}{m^2}\right)\right] + I_B(T,m)\,. \tag{15.16}$$

where

$$I_B(T,m) \equiv 2\frac{\partial}{\partial m^2} J_B(T,m) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{E_k}\frac{1}{e^{\beta E_k}-1} \tag{15.17}$$

## Perturbative Pressure Correction

- In the limit $x \to 0$, the free propagator $G_{\rm free}(x)$ diverges
- Because small distance corresponds to high frequency, this is again an example for a UV-divergence
- For now, note that the divergence is absent for $m = 0$ where for D=3

$$\lim_{m\to 0} G_{\rm free}(0) = I_B(T,0) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{k}\frac{1}{e^{\beta k}-1} = \frac{T^2\zeta(2)}{2\pi^2} \tag{15.18}$$

- Therefore, for a massless scalar field in 3+1 dimensions with $\phi^4$ interaction we have

$$p_{(1)} = p_{\rm free} - 3\lambda G^2_{\rm free}(0) = \frac{\pi^2 T^4}{90} - \lambda\frac{T^4}{48} \tag{15.19}$$

# Free vs. Full Propagator

- In perturbation theory, we consider expectation values w.r.t. the free action $S_0$
- That is how we defined the free propagator (15.2)
- However, in the *full* QFT, we can also consider the two-point function

$$G(x) \equiv \frac{\int \mathcal{D}\phi e^{-S_0 - S_I} \phi(x)\phi(0)}{Z} \tag{15.20}$$

- This is referred to as the *full* (or resummed) propagator, because it corresponds to an infinite number of terms in perturbation theory:

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\int \mathcal{D}\phi e^{-S_0} \phi(x)\phi(0)(-S_I)^n}{Z} = G_{\rm free} - \langle \phi(x)\phi(0) S_I \rangle_{\rm conn.} + \ldots \tag{15.21}$$

# Free vs. Full Propagator

- Similar to the pressure, we can calculate *approximations* to the full propagator through perturbation theory
- The perturbative series for the propagator can be written in terms of (connected) Feynman diagrams with two "external" legs
- Some of these infinitely many diagrams have a simple enough structure that we can calculate *all* of these
- Resumming the infinite series of diagrams that are easy to calculate, we will get an approximation for the *full* propagator
- This approach ("resummation") is aiming for *non-perturbative* results and (depending on the variation used) can be quite successful