Propagator in Scalar Field Theory II

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Fall 2020

Review

- In lecture 15, we introduced the free propagator as

$$G_{\rm free}(x) = \frac{\int \mathcal{D}\phi e^{-S_0}\phi(x)\phi(0)}{Z_{\rm free}} \tag{16.1}$$

- Let us concentrate on zero-temperature first. In Eq. (15.9) we found

$$\lim_{T\to 0} G_{\rm free}(x) = \int_K \frac{e^{iK\cdot x}}{K^2+m^2} \tag{16.2}$$

- In this lecture, we will discuss and interpret the free propagator result

Momentum Space Propagator

- Eq. (16.2) is in the form of a Fourier-transform
- The free propagator in *momentum space* is therefore

$$\tilde{G}_{\rm free}(K) = \frac{1}{K^2 + m^2}\,, \tag{16.3}$$

where $K^2 = \omega_n^2 + \vec{k}^2$ and $\omega_n = 2\pi n T$ are the Matsubara frequencies with $n = 0, \pm 1, \pm 2, \ldots$

- K^2 has a Lorentz-invariant form if we write

$$K^2 = -(i\omega_n)^2 + \vec{k}^2\,. \tag{16.4}$$

- We may try to perform an *analytic continuation* to Minkowski 4-momentum $k^\mu = \left(k^0, \vec{k}\right)$

Analytic Continuation

- We will analytically continue the (discrete) Matsubara frequencies ω_n to real momenta k^0 as

$$i\omega_n \to k^0 + i0^+ \,, \tag{16.5}$$

where 0^+ denotes a small regulator that is taken to zero at the end

- In the case of the propagator, this prescription leads to the *retarded Green's function* (cf. 1701.01554, chapter 8):

$$\begin{aligned} \tilde{G}_{\rm R,free}(k^0,\vec{k}) &= \tilde{G}_{\rm free}(-ik^0+0^+,\vec{k})\,, \\ &= \frac{1}{-(k^0+i0^+)^2+\vec{k}^2+m^2}\,, \\ &= \frac{1}{k_\mu k^\mu + m^2 - i0^+{\rm sign}(k^0)}\,. \end{aligned} \tag{16.6}$$

Analytic Continuation

This result can further be rewritten as

$$\begin{aligned}\tilde{G}_{\rm R,free}(k^0,\vec{k}) &= \mathbb{P}\left[\frac{1}{k_\mu k^\mu + m^2}\right] + \frac{i0^+ {\rm sign}(k^0)}{(k_\mu k^\mu + m^2)^2 + 0^{+2}}\,,\\ &= \mathbb{P}\left[\frac{1}{k_\mu k^\mu + m^2}\right] + \frac{i\pi}{2E_k}\left[\delta(k^0 - E_k) - \delta(k^0 + E_k)\right]\,,\end{aligned}$$

where $E_k^2 = m^2 + \vec{k}^2$, $\mathbb{P}$ denotes the principal value and the representation

$$\frac{0^+}{x^2 + (0^+)^2} \to \pi\delta(x)\,, \tag{16.7}$$

for the Dirac-delta function has been used

Quasi-Particle Interpretation

- The real and imaginary part of the retarded Green's function are not independent, they are related by a generalization of the optical theorem
- Let us concentrate on the imaginary part, which we call *spectral function*:

$$\tilde{\rho}_{\rm free}(k^0,\vec{k}) = {\rm Im}\,\tilde{G}_{\rm R,free}(k^0,\vec{k}) = \frac{\pi}{2E_k}\left[\delta(k^0-E_k)-\delta(k^0+E_k)\right]\,. \tag{16.8}$$

- The spectral function is sharply peaked at energies $k^0=\pm E_k$
- While we have been doing QFT, one of these looks just like a classical particle with localized energy $k^0=E_k$; we will call it a "quasiparticle"
- The other is stranger: taken at face value, it's a particle with *negative* energy; we will call it an *anti*-quasiparticle

Quasiparticle Dispersion Relations

- The quasi-particle structure of the spectral function suggests that the relevant excitations in free quantum field theory are particle-like
- The quasi-particle and its anti-particle in free-field theory fulfill

$$(k^0)^2 = k^2 + m^2 \,, \tag{16.9}$$

  which we call a *dispersion relation*
- The dispersion relation is often measurable in practice, and can be used to infer properties of quasiparticles
- In particular, the dispersion relation (16.9) implies that the quasiparticles have mass m
- Also, since (16.9) does not have any imaginary part, the quasi-particle lifetime is infinite — they are unconditionally stable in free field theory