Propagator in Scalar Field Theory III

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Review

- In lecture 15, we introduced the full propagator

$$G(x) = \frac{\int \mathcal{D}\phi e^{-S_0 - S_I} \phi(x)\phi(0)}{Z} . \tag{17.1}$$

- To first order perturbative approximation, the full propagator is

$$G_{(1)} = G_{\rm free} - \langle \phi(x)\phi(0) S_I \rangle . \tag{17.2}$$

  and the partition function is

$$Z_{(1)} = Z_{\rm free} - \langle S_I \rangle . \tag{17.3}$$

- In this lecture, we will discuss the perturbative propagator as well as resummations thereof

Perturbative Correction

- The first-order correction term is

$$\langle \phi(x)\phi(0)S_I\rangle = \lambda \int_y \langle \phi(x)\phi(0)\phi^4(y)\rangle \tag{17.4}$$

- Using Wick's theorem and only keeping connected diagrams, this becomes

$$\begin{aligned} \langle \phi(x)\phi(0)S_I\rangle &= 12\lambda \int_y \langle \phi(x)\phi(y)\rangle\langle \phi(y)\phi(0)\rangle\langle \phi^2(y)\rangle\,, \\ &= 12\lambda G_{\rm free}(0)\int_y G_{\rm free}(x-y)G_{\rm free}(y) \end{aligned} \tag{17.5}$$

Perturbative Correction

- Using (16.2), as well as $\int_y e^{iy\cdot(P-K)} = \delta(P-K)$ we have

$$\lim_{T\to 0}\langle\phi(x)\phi(0)S_I\rangle = 12\lambda G_{\rm free}(0)\int_K \frac{e^{iK\cdot x}}{(K^2+m^2)^2} \tag{17.6}$$

- We therefore find

$$\begin{aligned}\lim_{T\to 0} G_{(1)}(x) &= \int_K \frac{e^{iK\cdot x}}{K^2+m^2} - 12\lambda G_{\rm free}(0)\int_K \frac{e^{iK\cdot x}}{(K^2+m^2)^2}\,,\\ &= \int_K e^{iK\cdot x}\left(\frac{1}{K^2+m^2} - \frac{12\lambda G_{\rm free}(0)}{(K^2+m^2)^2}\right)\end{aligned} \tag{17.7}$$

Perturbative Correction

- Perturbative result in momentum space:

$$\tilde{G}_{(1)}(K) = \frac{1}{K^2+m^2} - \frac{12\lambda G_{\rm free}(0)}{(K^2+m^2)^2}\,. \tag{17.8}$$

- We have calculated $G_{\rm free}(0)$ in lecture 15 in dim-reg, cf. (15.16)
- Eq. (17.8) has a simple structure; it looks like the start of a geometric series
- By explicitly considering $\tilde{G}_{(2)}, \tilde{G}_{(3)}, \ldots$ one indeed finds that

$$\tilde{G}(K) = \frac{1}{K^2+m^2+12\lambda G_{\rm free}(0)} + \mathcal{O}(\lambda^2)\,, \tag{17.9}$$

which is valid up to $2^{\rm nd}$ order in perturbation theory

Self-energy

- If we allow for an arbitary function $\tilde{\Pi}(K)$, we can represent the full propagator in momentum space as

$$\tilde{G}(K) = \frac{1}{K^2 + m^2 + \tilde{\Pi}(K)} \,. \tag{17.10}$$

- To lowest order in perturbation theory, we have

$$\tilde{\Pi}(K) = 12\lambda G_{\rm free}(0) \,, \tag{17.11}$$

  e.g. just a constant
- We call Π the *self-energy* of the scalar field ϕ
- The self-energy is a central object in quantum field theory, and contains an enourmous amount of information

Perturbative renormalization of the Self-Energy

- Using (15.16) for $G_{\rm free}(0)$, the perturbative self-energy to first order is

$$\tilde{\Pi}_{(1)} = -\frac{3m^2\lambda}{4\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m^2}\right)\right] + 12\lambda I_B(T,m) \qquad (17.12)$$

- The result is divergent when letting $\varepsilon \to 0$
- However, note that $\tilde{\Pi}$ appears in the propagator only as $m^2 + \tilde{\Pi}$
- Next, realize that m^2 is just a parameter in the Lagrangian; we may add a counterterm similar to our renormalization program for the cosmological constant:

$$m^2 \to m^2_{\rm phys} + \delta m^2 \,. \qquad (17.13)$$

Perturbative renormalization of the Self-Energy

- We can *renormalize* $m^2 + \tilde{\Pi}_{(1)}$ by choosing a divergent mass-counterterm; in $\overline{\rm MS}$:

$$\delta m^2 = \frac{3 m^2_{\rm phys}\lambda}{4\pi^2 \varepsilon} + \mathcal{O}(\lambda^2) \tag{17.14}$$

- The resulting combination is finite to leading order in perturbation theory

$$\begin{aligned} m^2 + \tilde{\Pi}_{(1)} &= m^2_{\rm phys} - \frac{3\lambda m^2_{\rm phys}}{4\pi^2} \ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m^2_{\rm phys}}\right) + 12\lambda I_B(T, m_{\rm phys})\,, \\ &= m^2_{\rm phys} + \tilde{\Pi}^{\rm ren}_{(1)} + \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{17.15}$$

- Note: there are remaining divergencies starting at $\mathcal{O}(\lambda^2)$, which will have to be cancelled by counterterms of the same order

Quasi-Particle Mass

- In lecture 16, we found that the pole of the analytically continued free propagator corresponded to the mass of the quasiparticle
- Let us do this exercise for the full propagator (17.10)
- To first order in perturbation theory, when letting $K^2 \to -k_0^2 + \vec{k}^2$ we have a propagator pole located at

$$k_0^2 = k^2 + m_{\rm phys}^2 + \tilde{\Pi}_{(1)}^{\rm ren} \,. \tag{17.16}$$

- This means the quasi-particle mass is no longer given by the parameter m in the Lagrangian; instead, the *effective* mass of the quasi-particle is

$$m_{\rm eff}(T) = m_{\rm phys} - \frac{3\lambda m_{\rm phys}}{8\pi^2} \ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m_{\rm phys}^2}\right) + \frac{6\lambda}{m_{\rm phys}} I_B(T, m_{\rm phys}) \tag{17.17}$$

Quasi-Particle Mass

- At first glance, it seems that the effective mass is temperature and renormalization scale dependent
- At zero temperature, we have $I_B(T, m_{\rm phys}) = 0$ and hence

$$m^2_{\rm eff}(T=0) = m^2_{\rm phys} - \frac{3\lambda m^2_{\rm phys}}{4\pi^2} \ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m^2_{\rm phys}}\right) \tag{17.18}$$

- However, $m^2_{\rm eff}$ is a measurable quantity, it **cannot** depend on $\bar{\mu}$
- The only way out is that the parameter $m_{\rm phys}$ depends on $\bar{\mu}$, so that

$$\bar{\mu}\frac{\partial m^2_{\rm eff}}{\partial \bar{\mu}} = 0\,. \tag{17.19}$$

Quasi-Particle Mass

- Putting $m_{\rm phys} \to m_{\rm phys}(\bar\mu)$, (17.19) implies

$$0 = \bar\mu \frac{\partial m^2_{\rm phys}(\bar\mu)}{\partial \bar\mu}\left[1 - \frac{3\lambda}{4\pi^2}\ln\left(\frac{\bar\mu^2 e^{-\frac{1}{2}}}{m^2_{\rm phys}}\right)\right] - \frac{3\lambda m^2_{\rm phys}(\bar\mu)}{2\pi^2} \qquad (17.20)$$

- In perturbation theory, we can maintain (17.19) if

$$\bar\mu \frac{\partial m^2_{\rm phys}(\bar\mu)}{\partial \bar\mu} = \frac{3\lambda m^2_{\rm phys}(\bar\mu)}{2\pi^2} + \mathcal{O}(\lambda^2) \qquad (17.21)$$

- We find that in order for the measurable quasi-particle mass to be independent from an arbitrary choice $\bar\mu$, the Lagrangian parameter $m_{\rm phys}$ has to depend on $\bar\mu$; in QFT lingo, the mass "runs"

Quasiparticle Mass

- A special case is $m_{\mathrm{phys}} = 0$
- In this case, using (15.18), we have

$$m_{\mathrm{eff}}(T) = \sqrt{12\lambda I_B(T,0)} = \sqrt{\lambda}T \tag{17.22}$$

- The effective quasi-particle mass is independent of $\bar{\mu}$, but depends on temperature
- We call this the *in-medium* mass, because even if the quasi-particle is massless at $T = 0$, it acquires an effective mass through interactions with the thermal medium
- Note that in-medium masses are generically unavoidable for any value of m_{phys}