Application: Thermal Phase Transitions I

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Review

- In previous lectures, we have discussed some of the basics of quantum field theory
- Let us now use the knowledge gained to apply QFT tools for a problem of physical interest: thermal phase transitions
- Prime examples for thermal phase transitions are the QCD phase transition in the early universe, the electroweak phase transition and Higgs mechanism in particle physics
- We do not yet have the tools to do a fully quantitative calculation of either of these (requires gauge fields, fermions, etc.)
- But we do have the tools to discuss thermal phase transitions qualitatively using scalar field theory

A Model Potential

- In previous lectures, we have considered QFTs with scalar field potentials of the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4\,. \tag{18.1}$$

- For these potentials, we had

$$\langle\phi(x)\rangle = 0\,, \quad \langle\phi(x)\phi(y)\rangle = G_{\rm free}(x-y)\,\ldots \tag{18.2}$$

- Higher order correlators given in terms of products of propagators via Wick's theorem

A Model Potential

- In previous lectures, we have considered QFTs with scalar field potentials of the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4\,. \tag{18.3}$$

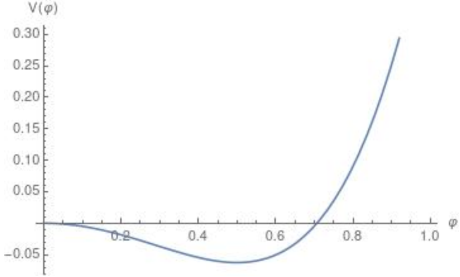
- Now let us flip the sign of m^2 and consider the potential

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \lambda\phi^4\,. \tag{18.4}$$

- We expect the quantum field theory to be well-behaved for $\lambda \neq 0$ because the ϕ^4 term makes integrals convergent despite the imaginary mass term

A Model Potential

Plotting the potential $V(\phi)$ for $\lambda = 1$, it looks like this:



The point $\phi = 0$ is classically unstable; this is a consequence of the negative m^2 term (imaginary mass always signals an instability)

A Model Potential

- Because for $\lambda = 0$ the potential is unstable, perturbation theory will not work as before
- In some sense, we would be expanding around the wrong "ground-state"
- However, it is easy to realize how this can be repaired: expand around the stable minimum of the potential instead of $\phi = 0$!

Mean Field

- We can formally separate the quantum field ϕ into a constant plus fluctuations:

$$\phi(x) = \bar{\phi} + \phi'(x)\,. \tag{18.5}$$

- We will refer to $\bar{\phi}$ as the “mean field”
- The path integral will become

$$Z = \int d\bar{\phi} \int \mathcal{D}\phi' e^{-\bar{S}[\bar{\phi}] - S'[\bar{\phi},\delta\phi]}\,, \tag{18.6}$$

where $S[\bar{\phi} + \phi'] = \bar{S}[\bar{\phi}] + S'[\bar{\phi}, \phi']$.
- Note that the integral over the mean field is a “regular” integral

Mean Field

- We can formally perform the integral over fluctuations ϕ' so that

$$Z = \int d\bar{\phi} e^{-\beta V V_{\rm eff}(\bar{\phi})}\,, \tag{18.7}$$

  where βV is the space-time volume of the Euclidean thermal cylinder
- and we call $V_{\rm eff}(\bar{\phi})$ the *effective potential*
- Note that in the large volume (low temperature) limit where $\beta V \to \infty$, the partition function is given by the minimum of the effective potential

$$Z = e^{-\beta V V_{\rm eff}\left(\bar{\phi}_{\rm min}\right) + \mathcal{O}(\ln(\beta V))}\,, \quad \left.\frac{dV_{\rm eff}}{d\bar{\phi}}\right|_{\bar{\phi}=\bar{\phi}_{\rm min}} = 0\,. \tag{18.8}$$

Mean Field

- To get some idea of what these new concepts mean, let us first simply ignore fluctuations
- This is a drastic approximation, but often not as bad as expected in capturing the correct physics
- If we simply ignore fluctuations, all we have is the mean field $\phi(x) = \bar{\phi}$ such that

$$Z = \int d\bar{\phi} e^{-\bar{S}[\bar{\phi}]}\,, \tag{18.9}$$

- Since $\phi(x)$ is constant, the kinetic term does not contribute to the action:

$$\bar{S}[\bar{\phi}] = \int d\tau \int d^3x V(\bar{\phi}) = \beta V V(\bar{\phi})\,, \tag{18.10}$$

where $V(\bar{\phi})$ is just (18.4) evaluated at $\phi = \bar{\phi}$.

Mean Field

- Thus we find in mean-field approximation

$$Z_{\mathrm{mf}} = \int d\bar{\phi} e^{-\beta V V(\bar{\phi})}, \qquad (18.11)$$

with $V(\bar{\phi})$ the *classical* potential

- Comparing (18.7) and (18.11), we find that the effective potential in the mean field approximation is given by the classical potential
- The classical potential (18.4) has a minimum located at

$$\bar{\phi}_{\mathrm{min}} = \pm \frac{m}{2\sqrt{\lambda}} \qquad (18.12)$$

so that within mean-field $\ln Z_{\mathrm{mf}} = \beta V \frac{m^4}{16\lambda} + \mathcal{O}(\ln(\beta V))$

Mean Field

- We may consider the expectation value of the field itself

$$\langle\phi(x)\rangle_{\rm full} = \frac{\int \mathcal{D}\phi e^{-S}\phi(x)}{Z}\,, \tag{18.13}$$

where the subscript "full" is a reminder that we are not doing perturbation theory

- In mean-field approximation, we have

$$\langle\phi(x)\rangle_{\rm mf} = \frac{\int d\bar{\phi}\phi e^{-\bar{S}}}{Z_{\rm mf}} = \bar{\phi} = \pm\frac{m}{2\sqrt{\lambda}}\,, \tag{18.14}$$

- In previous lectures, we found that when $T \gg m$ we can ignore m, such that we expect $\langle\phi(x)\rangle = 0$

Thermal Phase Transitions

- We conclude that in-between the zero-temperature case where $\langle\phi(x)\rangle \neq 0$, and the high temperature case where $\langle\phi(x)\rangle = 0$ there must be some sort of transition
- We call this a thermal phase transition
- and $\langle\phi(x)\rangle_{\rm full}$ is the order parameter of the transition