Application: Thermal Phase Transitions II

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Fall 2020

Review

- In lecture 18, we considered QFT with a scalar field potential

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \lambda\phi^4\,. \tag{19.1}$$

- Using the mean-field approximation, we found hints that with this potential, the theory contains a thermal phase transition
- In lecture 18, we found that the phase was controlled by the minimum of the effective potential $V_{\rm eff}$

$$Z = \int d\bar{\phi} e^{-\beta V V_{\rm eff}(\bar{\phi})}\,, \tag{19.2}$$

  where in the mean-field approximation $V_{\rm eff}(\bar{\phi}) = V(\bar{\phi})$
- In this lecture, we will put our study of this potential on more rigorous footing

Beyond Mean Field

- We again start by decomposing the scalar field into a constant plus fluctuations:

$$\phi(x) = \bar{\phi} + \phi'(x)\,. \tag{19.3}$$

where by construction $\int_x \phi'(x) = 0$.

- Using this decomposition, the potential (19.1) becomes

$$\begin{aligned} V[\bar{\phi} + \phi'] &= V[\bar{\phi}] + \left(-m^2\bar{\phi} + 4\lambda\bar{\phi}^3\right)\phi'\,, \\ &\quad + \frac{1}{2}\left(-m^2 + 12\lambda\bar{\phi}^2\right)\phi'^2 + 4\lambda\bar{\phi}\phi'^3 + \lambda\phi'^4\,, \end{aligned} \tag{19.4}$$

- Since the potential enters the action as $\int_x V(\phi)$, the term linear in ϕ' drops out

Beyond Mean Field

- After the decomposition, we have for the partition function

$$Z = \int d\bar{\phi} e^{-\beta V V(\bar{\phi})} \int \mathcal{D}\phi' e^{-S_0'[\bar{\phi}] - S_I'[\bar{\phi}]}\,, \tag{19.5}$$

where

$$S_0'[\bar{\phi}] = \frac{1}{2}\int_x \left[\partial_a \phi' \partial_a \phi' + \left(-m^2 + 12\lambda\bar{\phi}^2\right)\phi'^2\right] \tag{19.6}$$

and

$$S_I'[\bar{\phi}] = \int_x \left[4\lambda\bar{\phi}\phi'^3 + \lambda\phi'^4\right]\,. \tag{19.7}$$

- Introducing the "effective mass" squared

$$m^2_{\rm eff}(\bar{\phi}) = -m^2 + 12\lambda\bar{\phi}^2\,, \tag{19.8}$$

the partition function (19.5) can be studied using perturbation theory

Beyond Mean Field

- If $\bar{\phi} = 0$, the calculation for the fluctuations reverts to the standard perturbation theory; we refer to this case as "symmetric phase"
- If $\bar{\phi} \neq 0$, the calculation is modified because of the presence of the non-zero expectation value for the field;we refer to this as the "broken phase"
- The theory will tell us which phase is realized for which temperature

Beyond Mean Field: Perturbation Theory

- Let us now calculate the effect from fluctuations on the effective potential in perturbation theory
- To leading order in perturbation theory, we ignore S_I and thus have to calculate

$$\int \mathcal{D}\phi' e^{-S_0'[\bar{\phi}]} \,. \tag{19.9}$$

- Since $S_0'[\bar{\phi}]$ is quadratic in ϕ', we can re-use our result from free field theory in lecture 9
- The only modification is that now the mass $m_{\rm eff}(\bar{\phi})$ depends on an external parameter $\bar{\phi}$

Beyond Mean Field: Perturbation Theory

- From equations (9.12), (10.2) we have

$$\int \mathcal{D}\phi' e^{-S_0'[\bar\phi]} = e^{-\frac{\beta V}{2}\int \frac{d^D k}{(2\pi)^D}\left[E_k + 2T\ln\left(1-e^{-\beta E_k}\right)\right]} = e^{\beta V p_{\rm free}(T)}\,, \tag{19.10}$$

  where $E_k^2 = \vec{k}^2 + m^2_{\rm eff}(\bar\phi)$.

- We have calculated the renormalized free pressure in $\overline{\rm MS}$ in lecture 11
- Using (11.16), to leading order in perturbation theory, the renormalized effective potential in (19.2) is given by

$$V^{\rm ren}_{\rm eff,0}(\bar\phi) = V(\bar\phi) - \frac{m^4_{\rm eff}(\bar\phi)}{64\pi^2}\ln\left(\frac{\bar\mu^2 e^{\frac{3}{2}}}{|m^2_{\rm eff}(\bar\phi)|}\right) + J_B(T, m_{\rm eff}(\bar\phi)) \tag{19.11}$$

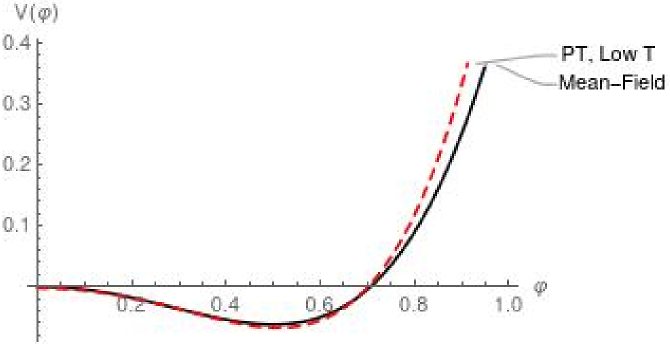
Beyond Mean Field: Perturbation Theory

- At zero temperature $J_B = 0$; choosing $\bar{\mu}^2 = m^2$, the perturbative effective potential becomes

$$V^{\rm ren}_{\rm eff,0}(\bar{\phi}) = V(\bar{\phi}) - \frac{m^4_{\rm eff}(\bar{\phi})}{64\pi^2} \ln\left(\frac{m^2 e^{\frac{3}{2}}}{|m^2_{\rm eff}(\bar{\phi})|}\right) \tag{19.12}$$

- For illustration, we can compare this to the mean-field approximation $V_{\rm mf}(\bar{\phi}) = V(\bar{\phi})$ for $\lambda = 1$

Low-temperature Effective Potential



At low temperature, there is not much change from the mean-field result

Beyond Mean Field: Perturbation Theory

- Next let's consider high temperature $T \gg m$
- We need an expansion of $J_B(T, m)$ in powers of m
- The leading order term was given in (11.5); the next-to-leading order term can be gleaned from Eqns. (15.17), (15.18); we find

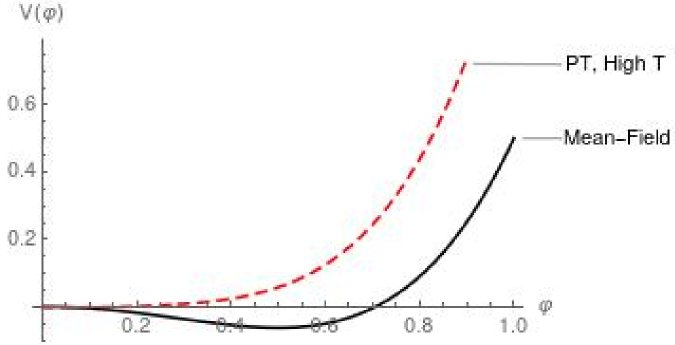
$$J_B(T, m) = -\frac{\pi^2 T^4}{90} + \frac{T^2 m^2}{24} + \dots \tag{19.13}$$

- $\bar{\phi}$-independent terms do not influence the shape of the potential; ignoring those, (19.13) leads to

$$V_{\rm eff,0}^{\rm ren}(\bar{\phi}) = \frac{1}{2}\left(-m^2 + \lambda T^2\right)\bar{\phi}^2 + \lambda\bar{\phi}^4 - \frac{m_{\rm eff}^4(\bar{\phi})}{64\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{|m_{\rm eff}^2(\bar{\phi})|}\right) \tag{19.14}$$

- Let us again visualize this result for $\bar{\mu} = m, \lambda = 1$

High-temperature Effective Potential



At high temperature, the symmetry is restored!

Symmetry restoration at high temperature

- At high temperature, the quadratic term in $V_{\rm eff}(\bar{\phi})$ becomes positive because λT^2 dominates over $-m^2$
- As a consequence, the minimum of the potential at high temperature is located at $\bar{\phi} = 0$
- Therefore, in the high temperature phase, the field expectation value $\langle \phi(x) \rangle_{\rm full} = 0$, and we are in the symmetric phase
- We can *estimate* the location of the phase transition between low-temperature and high-temperature phase in perturbation theory

Estimate of the transition temperature

- To estimate the transition temperature in perturbation theory, start with the derivative of the potential $\frac{dV_{\rm eff}(\bar\phi)}{d\bar\phi}$ with $\bar\mu^2 = m^2$
- Since for the symmetric phase $\bar\phi = 0$, we can evaluate

$$\left.\frac{dV_{\rm eff}(\bar\phi)}{d\bar\phi}\right|_{\bar\phi=0} = m^2\left(-1+\frac{3\lambda}{4\pi^2}\right)+\lambda T^2\,. \tag{19.15}$$

- Since $\bar\phi = 0$ must be a minimum of the potential, this leads to

$$T_c^2 = m^2\left(\frac{1}{\lambda}-\frac{3}{4\pi^2}\right) \tag{19.16}$$

- Since we are doing perturbation theory for $\lambda \ll 1$, our estimate is

$$T_c = \frac{m}{\sqrt{\lambda}}\,. \tag{19.17}$$