Vertex Corrections

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Review

- In lecture 17, we considered the full propagator

$$G_{\rm full}(x) \equiv \frac{\int \mathcal{D}\phi e^{-S}\phi(x)\phi(0)}{Z}\,. \tag{20.1}$$

  in perturbation theory
- The full propagator is a *two-point* function
- In this lecture, we will discuss the full *four-point* function

$$\Gamma_4(x_1,x_2,x_3,x_4) \equiv \frac{\int \mathcal{D}\phi e^{-S}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}{Z} \tag{20.2}$$

  in perturbation theory

Leading order

- As before, we are only interested in the *connected* four-point function
- The minimum number of vertices to connect four fields ϕ is one
- Therefore, the leading order in perturbation theory is first order,

$$\Gamma^{(1)}_{4,\mathrm{conn.}}(x_1,x_2,x_3,x_4) = \frac{\int \mathcal{D}\phi e^{-S_0}\left(-S_I\right)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}{Z_{(1)}} \tag{20.3}$$

- Counting the number of possible contractions, one finds

$$\Gamma^{(1)}_{4,\mathrm{conn.}} = -4!\lambda \int_x G_{\mathrm{free}}(x-x_1)G_{\mathrm{free}}(x-x_2)G_{\mathrm{free}}(x-x_3)G_{\mathrm{free}}(x-x_4)\,. \tag{20.4}$$

Leading order

- We can use the momentum-space representation of the free propagator (15.9) to write

$$
\begin{aligned}
\Gamma^{(1)}_{4,\mathrm{conn.}} &= -4!\lambda \int_{P_1,P_2,P_3,P_4} \tilde{G}_{\mathrm{free}}(P_1)\tilde{G}_{\mathrm{free}}(P_2)\tilde{G}_{\mathrm{free}}(P_3)\tilde{G}_{\mathrm{free}}(P_4) \\
&\quad \times (2\pi)^4\delta(P_1+P_2+P_3+P_4)e^{iP_1\cdot x_1+iP_2\cdot x_2+iP_3\cdot x_3+iP_4\cdot x_4}\,.
\end{aligned}
$$

- The δ-function ensures momentum conservation at the vertex
- The four propagators are a consequence of the four "legs" of the vertex

Leading order

- We can use the momentum-space representation of the free propagator (15.9) to write

$$\begin{aligned}\Gamma^{(1)}_{4,\mathrm{conn.}} &= -4!\lambda \int_{P_1,P_2,P_3,P_4} \tilde{G}_{\mathrm{free}}(P_1)\tilde{G}_{\mathrm{free}}(P_2)\tilde{G}_{\mathrm{free}}(P_3)\tilde{G}_{\mathrm{free}}(P_4)\\ &\quad\times(2\pi)^4\delta(P_1+P_2+P_3+P_4)e^{iP_1\cdot x_1+iP_2\cdot x_2+iP_3\cdot x_3+iP_4\cdot x_4}\,.\end{aligned}$$

- Similar to the case of the propagator, we may define a *full* momentum-space four point function

$$\tilde{\Gamma}_4(P_1,P_2,P_3,P_4)=\int_{x_1,x_2,x_3,x_4} e^{-iP_1\cdot x_1-iP_2\cdot x_2-iP_3\cdot x_3-iP_4\cdot x_4}\Gamma_4(x_1,x_2,x_3,x_4) \tag{20.5}$$

- $\tilde{\Gamma}_4(P_1,P_2,P_3,P_4)$ must have four propagators $\tilde{G}(P)$, and must contain a δ function to ensure momentum conservation

Leading order

- From

$$\Gamma^{(1)}_{4,\text{conn.}} = -4!\lambda \int_{P_1,P_2,P_3,P_4} \tilde{G}_{\text{free}}(P_1)\tilde{G}_{\text{free}}(P_2)\tilde{G}_{\text{free}}(P_3)\tilde{G}_{\text{free}}(P_4)$$
$$\times(2\pi)^4\delta(P_1+P_2+P_3+P_4)e^{iP_1\cdot x_1+iP_2\cdot x_2+iP_3\cdot x_3+iP_4\cdot x_4},$$

  we can define an "amputated" connected vertex function, which to leading order is given by

$$\tilde{\Gamma}^{(1)}_4\Big|_{\text{conn.,amp.}}(P_1,P_2,P_3,P_4) = -4!\lambda \qquad (20.6)$$

- Here "amputated" means neglect the four "external leg" propagators $\tilde{G}_{\text{free}}(P_1)\tilde{G}_{\text{free}}(P_2)\tilde{G}_{\text{free}}(P_3)\tilde{G}_{\text{free}}(P_4)$ and $(2\pi)^4\delta(P_1+P_2+P_3+P_4)$

Next-to-Leading Order

- At Next-to-Leading order (NLO) in perturbation theory, we have

$$\Gamma^{(2)}_{4,\mathrm{conn.}} = \Gamma^{(1)}_{4,\mathrm{conn.}} + \frac{1}{2}\frac{\int \mathcal{D}\phi e^{-S_0} \int_{x,y} \phi^4(x)\phi^4(y)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}{Z_{(2)}}$$

- Let's pick $\phi(x_1)$ first: there are 2 times 4 "legs" in S_I^2 to attach to, and we pick $\phi(x)$
- Next pick $\phi(x_2)$: there are 3 "legs" $\phi(x)$ to attach to if we want $\phi(x_2)$ connect to the same vertex as $\phi(x_1)$
- Next pick one of the remaining $\phi(x)$ and attach to one of the 4 $\phi(y)$, and then pick the last $\phi(x)$ and attach to the 3 $\phi(y)$
- Finally, attach $\phi(x_3)$ to one of the 2 remaining $\phi(y)$ and $\phi(x_4)$ to the last $\phi(y)$
- The overall multiplicative factor is

$$\frac{1}{2} \times 2 \times 4 \times 3 \times 4 \times 3 \times 2 = \frac{(4!)^2}{2}\,. \tag{20.7}$$

Next-to-Leading Order

- We made a choice to attach $\phi(x_2)$ at the same vertex as $\phi(x_1)$; two other choices are possible that lead to the same amputated connected four-point function where the labels x_1, x_2, x_3, x_4 are exchanged
- We can again define an amputated connected four-point function in momentum space
- Performing all the contractions and converting to momentum space, we obtain

$$\begin{aligned}\tilde{\Gamma}^{(2)}_{4,\mathrm{conn.,amp.}} &= \tilde{\Gamma}^{(1)}_{4,\mathrm{conn.,amp.}} \\ &+\frac{(4!\lambda)^2}{2}\int_K \frac{1}{K^2+m^2}\frac{1}{(P_1+P_2-K)^2+m^2} \\ &+\text{two others}\,. \end{aligned} \tag{20.8}$$

Next-to-Leading Order

- It's instructive to consider the vertex at vanishing external momentum such that

$$\tilde{\Gamma}^{(2)}_{4,\text{conn.,amp.}}(P=0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{2}\int_K \frac{1}{(K^2+m^2)^2} \quad (20.9)$$

- At zero temperature, we can evaluate the remaining integral using (10.5), (10.1), (10.16):

$$\Phi(m,D,A) = \mu^{D-4}\int_K (K^2+m^2)^{-A} = \frac{\mu^{D-4}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(A-\frac{D}{2})}{\Gamma(A)}(m^2)^{-A+\frac{D}{2}} \quad (20.10)$$

- We find in dim-reg for $D = 4-2\varepsilon$:

$$\tilde{\Gamma}^{(2)}_{4,\text{conn.,amp.}}(P=0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{2}\frac{1}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon)\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}. \quad (20.11)$$

Renormalizing the Coupling Constant

- The connected amputated four-point vertex is divergent for $\varepsilon \to 0$
- Expanding in powers of ε, we find

$$\tilde{\Gamma}^{(2)}_{4,\mathrm{conn.,amp.}}(P=0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] + \mathcal{O}(\varepsilon)\,. \tag{20.12}$$

- In order for Γ_4 to be finite, we need to *renormalize the coupling*
- In analogy with the mass renormalization in lecture 17, we re-define the parameter λ in the Lagrangian as

$$\lambda = \lambda_{\mathrm{phys}} + \delta\lambda \tag{20.13}$$

with $\delta\lambda = \mathcal{O}(\lambda^2_{\mathrm{phys}})$

Renormalizing the Coupling Constant

- In the $\overline{\mathrm{MS}}$ scheme we choose

$$\delta\lambda = \frac{9\lambda_{\rm phys}^2}{4\pi^2\varepsilon} \tag{20.14}$$

- The renormalized connected amputated four-point vertex then becomes

$$\tilde{\Gamma}^{(2)}_{4,{\rm conn.,amp.,ren.}}(P=0) = -4!\lambda_{\rm phys} + \frac{9\lambda_{\rm phys}^2}{4\pi^2}\left[\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right]\,. \tag{20.15}$$

- Note that $\tilde{\Gamma}^{(2)}_{4,{\rm conn.,amp.,ren.}}(P=0)$ is a measurable quantity; it cannot depend on the arbitrary scale $\bar{\mu}$

Renormalizing the Coupling Constant

- Because $\tilde{\Gamma}^{(2)}_{4,\mathrm{conn.,amp.,ren.}}(P=0)$ cannot depend on $\bar{\mu}$, we must have

$$\bar{\mu}\frac{\partial\tilde{\Gamma}^{(2)}_{4,\mathrm{conn.,amp.,ren.}}(P=0)}{\partial\bar{\mu}}=0\,. \tag{20.16}$$

- However, we have found an explicitly $\bar{\mu}$ dependence in perturbation theory
- The only way out is that λ_{phys} is in fact $\bar{\mu}$ dependent
- Using $\lambda_{\mathrm{phys}}=\lambda_{\mathrm{phys}}(\bar{\mu})$, (20.16) implies

$$0=\bar{\mu}\frac{\partial\lambda_{\mathrm{phys}}(\bar{\mu})}{\partial\bar{\mu}}\left[1-\frac{3\lambda_{\mathrm{phys}}}{16\pi^2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right]-\frac{3\lambda^2_{\mathrm{phys}}}{16\pi^2} \tag{20.17}$$

Renormalizing the Coupling Constant

- To leading order in perturbation theory therefore

$$\bar{\mu}\frac{\partial \lambda_{\rm phys}(\bar{\mu})}{\partial \bar{\mu}} = \frac{3\lambda^2_{\rm phys}(\bar{\mu})}{16\pi^2} + \mathcal{O}(\lambda^3_{\rm phys}) \tag{20.18}$$

- The coupling constant changes with $\bar{\mu}$, it therefore "runs"
- We will explore the consequence (20.18) in the following lecture