Renormalization Group

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Review

- Divergencies in QFT require renormalization
- Renormalized expressions generically involve arbitrary scale $\bar{\mu}$
- Physical observables cannot depend on arbitrary scale, but parameters in Lagrangian can
- In lecture 17, independence of physical quasi-particle mass led us to

$$\bar{\mu}\frac{\partial m^2_{\rm phys}(\bar{\mu})}{\partial\bar{\mu}} = \frac{3\lambda_{\rm phys}(\bar{\mu}) m^2_{\rm phys}(\bar{\mu})}{2\pi^2} + \mathcal{O}(\lambda^2_{\rm phys}) \tag{21.1}$$

- In lecture 20, independence of physical vertex led us to

$$\bar{\mu}\frac{\partial \lambda_{\rm phys}(\bar{\mu})}{\partial\bar{\mu}} = \frac{3\lambda^2_{\rm phys}(\bar{\mu})}{16\pi^2} + \mathcal{O}(\lambda^3_{\rm phys}) \tag{21.2}$$

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- Eqns. (21.1), (21.2) represent the dependence of Lagrangian parameters on the arbitrary scale $\bar{\mu}$
- Since these imply that $\lambda_{\rm phys}, m_{\rm phys}$ change with μ we call $\lambda_{\rm phys}$ the *running coupling constant* and $m_{\rm phys}$ the *running mass*
- It is customary to introduce the notation

$$\beta(\lambda) \equiv \bar{\mu}\frac{\partial \lambda(\bar{\mu})}{\partial \bar{\mu}}\,, \quad \gamma_m(\lambda) \equiv \bar{\mu}\frac{\partial \ln m^2(\bar{\mu})}{\partial \bar{\mu}}\,, \tag{21.3}$$

the so-called β and γ functions

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- Let us consider a physically measurable object such as the pressure in QFT
- When calculating the pressure in QFT, we typically encounter divergencies, requiring renormalization
- After renormalization, the pressure depends on parameters in the Lagrangian, e.g.

$$p_{\rm ren} = p_{\rm ren}(\bar\mu, \lambda_{\rm phys}(\bar\mu), m_{\rm phys}(\bar\mu)) \tag{21.4}$$

- However, since the pressure is a physical observable, choosing a different scale $\bar\mu \to \bar\mu'$ must give the same pressure:

$$p_{\rm ren}(\bar\mu, \lambda_{\rm phys}(\bar\mu), m_{\rm phys}(\bar\mu)) = p_{\rm ren}(\bar\mu', \lambda_{\rm phys}(\bar\mu'), m_{\rm phys}(\bar\mu')) \tag{21.5}$$

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- Put differently, a physical observable is invariant under changes of the renormalization scale
- We call this "renormalization group invariant"
- For the case of the pressure, renormalization group invariance implies

$$\bar{\mu}\frac{dp_{\rm ren}(\bar{\mu},\lambda_{\rm phys}(\bar{\mu}),m_{\rm phys}(\bar{\mu}))}{d\bar{\mu}}=0 \tag{21.6}$$

- Here $\frac{d}{d\bar{\mu}}$ is a *total* derivative
- We can use the chain rule to split it up:

$$\left[\bar{\mu}\frac{\partial}{\partial\bar{\mu}}+\beta\frac{\partial}{\partial\lambda_{\rm phys}}+\gamma_m m^2_{\rm phys}\frac{\partial}{\partial m^2_{\rm phys}}\right]p_{\rm ren}(\bar{\mu},\lambda_{\rm phys},m_{\rm phys})=0\,. \tag{21.7}$$

- Let's do an example for the entropy density $s_{\rm ren}=\frac{\partial p_{\rm ren}}{\partial T}$ in order to avoid issues with the cosmological constant; using $p=p_{\rm free}-3\lambda G^2_{\rm free}(0)$, we have

$$s_{(1)}=-\frac{\partial J_B(T,m)}{\partial T}-6\lambda G_{\rm free}(0)\frac{\partial I_B(T,m)}{\partial T} \qquad (21.8)$$

- Now since only $G_{\rm free}(0)$ carries a $\bar\mu$-dependence

$$\bar\mu\frac{\partial s_{(1)}}{\partial\bar\mu}=-6\lambda\frac{\partial I_B(T,m)}{\partial T}\bar\mu\frac{\partial G_{\rm free}(0)}{\partial\bar\mu} \qquad (21.9)$$

- Using Eq.(15.16) for $G_{\rm free}(0)$, RG-invariance implies

$$6\lambda\frac{\partial I_B(T,m)}{\partial T}\frac{m^2}{8\pi^2}=-\left[\beta\frac{\partial}{\partial\lambda_{\rm phys}}+\gamma_m m^2_{\rm phys}\frac{\partial}{\partial m^2_{\rm phys}}\right]s_{(1)} \qquad (21.10)$$

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- To lowest order in perturbation theory, $\lambda = \lambda_{\rm phys}$, $m = m_{\rm phys}$
- Furthermore, (21.2) implies $\beta = \mathcal{O}(\lambda_{\rm phys}^2)$, so the term $\beta\partial_\lambda$ does not contribute to order $\mathcal{O}(\lambda_{\rm phys})$
- Moreover, (21.1) implies $\gamma_m = \mathcal{O}(\lambda_{\rm phys})$, so to first order in perturbation theory

$$
\begin{aligned}
\frac{\partial I_B}{\partial T}\frac{6\lambda_{\rm phys} m_{\rm phys}^2}{8\pi^2} = -\gamma_m m_{\rm phys}^2 \frac{\partial}{\partial m_{\rm phys}^2} s_{(1)} &= \gamma_m m_{\rm phys}^2 \frac{\partial}{\partial m_{\rm phys}^2}\frac{\partial J_B}{\partial T}\,,\\
&= \gamma_m \frac{m_{\rm phys}^2}{2}\frac{\partial I_B}{\partial T}\,,\\
&= \frac{3\lambda_{\rm phys}}{2\pi^2}\frac{m_{\rm phys}^2}{2}\frac{\partial I_B}{\partial T}\,.
\end{aligned}
$$

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- The renormalization group implies consistency conditions that can be used to check perturbative results
- More importantly, RG implies evolution equations such as (21.1), (21.2) that can be used to solve for $\lambda_{\rm phys}, m_{\rm phys}$

The Running Coupling in ϕ^4 Theory

- To leading order in perturbation theory, the renormalized coupling constant in the Lagrangian fulfills (21.2)
- Ignoring higher order perturbative corrections, we may rewrite (21.1) as

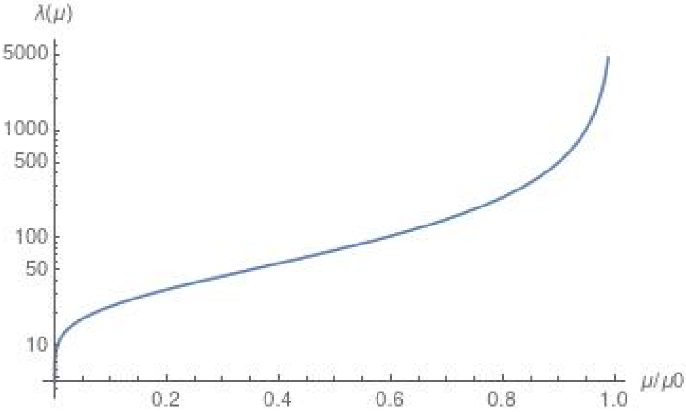
$$\frac{3}{16\pi^2} = \frac{1}{\lambda_{\rm phys}^2}\frac{\partial \lambda_{\rm phys}}{\partial \ln \bar{\mu}} = -\frac{\partial \lambda_{\rm phys}^{-1}}{\partial \ln \bar{\mu}} \qquad (21.11)$$

- This can be integrated w.r.t $\bar{\mu}$ to find

$$\lambda_{\rm phys}(\bar{\mu}) = \frac{\frac{16\pi^2}{3}}{\ln\left(\frac{\mu_0}{\bar{\mu}}\right)}\,, \qquad (21.12)$$

where μ_0 is an integration constant with $[\mu_0] = 1$

Running Coupling in ϕ^4 Theory



The Running Coupling in ϕ^4 Theory

- Renormalized coupling $\lambda_{\rm phys}$ is small for small energies
- $\lambda_{\rm phys}(\bar{\mu})$ grows as a function of energy scale $\bar{\mu}$ (the coupling is "running")
- The increase of $\lambda_{\rm phys}(\bar{\mu})$ is a result of the positive sign of the β function; a negative β function would lead to a *decreasing* $\lambda_{\rm phys}$
- Positive β function implies that the physical coupling of the theory gets stronger at short distances; this is a problem for thinking about the *continuum limit* of the field theory
- Theories with negative β function behave the opposite way: the coupling is small at short distances, and the theory is weakly coupled in the continuum limit; such theories are called *asymptotically free*

The Running Coupling in ϕ^4 Theory

- Strange result for ϕ^4 theory: $\lambda_{\rm phys} \to \infty$ for a *finite* energy scale $\bar{\mu} = \mu_0$
- We call the scale μ_0 where the coupling diverges the *Landau pole*
- Since the β function was calculated in perturbation theory, we cannot trust that our analysis correctly identifies the large λ behavior of the theory; so we don't know for sure if the Landau pole is there or not
- If there truly is a Landau pole in the theory, there is a minimum length scale $\propto \mu_0^{-1}$ below which the theory does not make any sense; the theory must be regarded as a cut-off dependent *effective theory*
- There are strong arguments suggesting that there is a Landau pole in QED