The Complex Scalar Field

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Review

- In lecture 7, the QFT partition function for a single real scalar field ϕ was given by

$$Z = \int \mathcal{D}\phi e^{-S_E} \,, \tag{22.1}$$

  where S_E is the classical Euclidean action

- The classical Euclidean action was constrained by containing only terms consistent with special relativity
- Restricting to second order in derivatives, we found for a real scalar field in lecture 8

$$S_E = \int_x \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right] \,. \tag{22.2}$$

- Let us now discuss what changes if we consider a *complex* scalar field

The Complex Scalar Field

- For a single *real* scalar field, invariance under Lorentz transformation restricts the classical action to (22.2)
- If we consider a *complex* scalar field, we need to generalize (22.2)
- Since physical observables such as the pressure do not have imaginary parts, we expect the partition function to be real
- The simplest way to enforce Z to be real is if the Euclidean action S_E is real
- If S_E has to be real, it must be built out of quadratic forms such as $\phi\phi^*$, where ϕ^* denotes the complex conjugate of ϕ
- If S_E has to be real and invariant under Lorentz transformations,

$$S_E = \int_x \left[\partial_a\phi\partial_a\phi^* + V(\sqrt{\phi\phi^*})\right] \tag{22.3}$$

  will work

The Complex Scalar Field

- It is instructive to consider a QFT for a complex scalar field that is close to the real scalar field case we have considered before
- For this reason, let us study the particular case

$$S_E = \int_s \left[\partial_a \phi \partial_a \phi^* + m^2 \phi \phi^* + 4\lambda \left(\phi\phi^*\right)^2\right] \tag{22.4}$$

  in the following
- Note that in addition to Lorentz invariance, the action (22.4) has an additional symmetry: it is invariant under the transformation

$$\phi(x) \to e^{i\alpha}\phi(x)\,, \quad \left(\phi^*(x) \to e^{-i\alpha}\phi^*(x)\right)\,, \tag{22.5}$$

  with arbitrary (but constant) α
- We will explore the consequence of this symmetry in future lectures

The Complex Scalar Field

- For now, let us aim at calculating the partition function for the complex scalar field
- Since ϕ is complex, we can separate it into a real and imaginary component

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) , \tag{22.6}$$

  with real ϕ_1, ϕ_2
- In terms of these components, the action (22.4) becomes

$$S_E = \int_x \left[\frac{1}{2}\partial_a\phi_1\partial_a\phi_1 + \frac{1}{2}\partial_a\phi_2\partial_a\phi_2 + \frac{m^2}{2}\phi_1^2 + \frac{m^2}{2}\phi_2^2 + \lambda\left(\phi_1^2+\phi_2^2\right)^2\right] . \tag{22.7}$$

- Similarly,

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S_E} . \tag{22.8}$$

The Complex Scalar Field

- When expressed in components, the action for the complex scalar field looks like two copies of a real scalar field
- The only coupling between the two copies is provided by the cross-term $2\lambda\phi_1^2\phi_2^2$
- Let's first inspect what happens if we drop the coupling entirely and consider the *free complex scalar field*

The Free Complex Scalar Field

- For the free complex scalar field, we have

$$S_E\big|_{\lambda=0} = S_0[\phi_1] + S_0[\phi_2]\,, \tag{22.9}$$

where $S_0[\phi] = \int_x \left[\frac{1}{2}\partial_a\phi\partial_a\phi + \frac{m^2}{2}\phi^2\right]$ is the free action for a real scalar field ϕ

- We find that the path integral factorizes:

$$Z_{\rm free} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S_0[\phi_1]-S_0[\phi_2]} = \int \mathcal{D}\phi_1 e^{-S_0[\phi_1]} \times \int \mathcal{D}\phi_2 e^{-S_0[\phi_2]} \tag{22.10}$$

- As a consequence, the partition function for the complex scalar field is given by

$$Z_{\rm free}^{\rm complex} = \left(Z_{\rm free}^{\rm real}\right)^2 \tag{22.11}$$

The Free Complex Scalar Field

- Since the pressure is defined as $p = \frac{\ln Z}{\beta V}$ we find for the pressure of the free complex scalar field

$$p_{\rm free}^{\rm complex} = \frac{\ln Z_{\rm free}^{\rm complex}}{\beta V} = 2\frac{\ln Z_{\rm free}^{\rm real}}{\beta V} = 2 p_{\rm free}^{\rm real} \tag{22.12}$$

- Using the $m = 0$ result (11.5) $p_{\rm free}^{\rm real,ren} = \frac{\pi^2 T^4}{90}$ for the renormalized pressure at temperature T for the real scalar field, we find

$$p_{\rm free}^{\rm complex,ren} = \frac{\pi^2 T^4}{45} \tag{22.13}$$

Degrees of Freedom

- Let's discuss the physics behind the result (22.13)
- For a free real scalar field, we found

$$p^{\mathrm{real}} = \frac{\pi^2 T^4}{90} \tag{22.14}$$

- For a free complex scalar field, we found

$$p^{\mathrm{complex}} = \frac{\pi^2 T^4}{45}\,, \tag{22.15}$$

  because it corresponds to *two* real scalar fields
- It's easy to generalize this: for N free real scalar fields, we will have

$$p^{(N)} = N \times \frac{\pi^2 T^4}{90}\,. \tag{22.16}$$

Degrees of Freedom

- Every free scalar field contributes $\frac{\pi^2 T^4}{90}$ to the pressure
- In the following we call this "one (bosonic) degree of freedom"
- Turning the argument around, we can use the pressure to *measure* the number of degrees of freedom in a system
- E.g. given a pressure at temperature T of $p(T)$, the number of degrees of freedom can be defined as

$$\mathrm{dof} = \frac{90p(T)}{\pi^2 T^4} \tag{22.17}$$

- The dof defined this way need not be integer
- The concept makes sense e.g. in cosmology, where dof changes depending on which constituents are relevant, e.g. $W^\pm, Z$ bosons, gluons, etc.

Degrees of Freedom

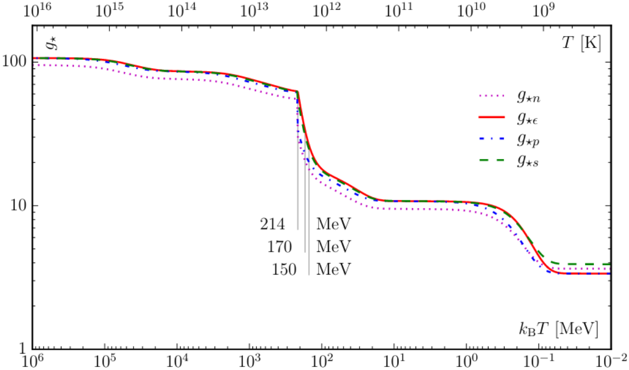

Figure 1. The evolution of the number density ($g_{\star n}$), energy density ($g_{\star \epsilon}$), pressure ($g_{\star p}$), and entropy density ($g_{\star s}$) as functions of temperature.

[https://arxiv.org/pdf/1609.04979.pdf]