Symmetries and Conserved Quantities

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Review

- In lecture 22, we discussed the action for the complex scalar field; in Minkowski space, we have

$$S = -\int_x \left[\partial_\mu \phi \partial^\mu \phi^* + m^2 \phi\phi^* + 4\lambda \left(\phi\phi^*\right)^2\right] . \qquad (23.1)$$

- We noted that the action has an additional symmetry under the transformation

$$\phi(x) \to e^{i\alpha}\phi(x) , \quad \left(\phi^*(x) \to e^{-i\alpha}\phi^*(x)\right) , \qquad (23.2)$$

  with arbitrary (but constant) α
- Let us now discuss the consequence of this symmetry

Classical Equations of Motion

- Let's write $S=\int_x \mathcal{L}$ with $\mathcal{L}(\phi,\phi^*)$ the Lagrangian density
- We can get the *classical* equations of motion from requiring S to be invariant under changes of ϕ: $S\rightarrow S'=S$ or $\delta S=0$
- Treating ϕ,ϕ^* as independent fields,

$$\delta S=\int_x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi+\frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^*+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\partial_\mu\phi^*\right]. \quad (23.3)$$

- Using the chain rule, we can write

$$\int_x\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi=\int_x\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right]-\int_x\delta\phi\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right], \quad (23.4)$$

and similarly for ϕ^*

Classical Equations of Motion

- The *classical* equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 = -m^2 \phi^* - 8\lambda(\phi\phi^*)\phi^* + \Box\phi^*\,, \quad (23.5)$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} = 0 = -m^2 \phi - 8\lambda(\phi\phi^*)\phi + \Box\phi\,. \quad (23.6)$$

- If we *use* these classical equations of motion, we get

$$\delta S = \int_x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}\delta\phi^*\right]\,. \quad (23.7)$$

- As long as the integrand is well-behaved at infinity, the integral of a total derivative vanishes
- We therefore have

$$\delta S = 0 \quad (23.8)$$

for *any* small variation $\delta\phi$ that's well-behaved at infinity

Symmetry and Conserved Quantities

- Now consider putting the system into a box; the equations of motion (23.5) still hold, and we find

$$\delta S_{\rm box}=\int_{\rm box} d^4x\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\phi^*\right]\,. \tag{23.9}$$

- For the symmetry (23.2), the action is invariant, hence $\delta S_{\rm box}=0$
- Plugging in $\delta\phi=i\alpha\phi(x)$ from (23.2), this gives

$$0=\alpha\int_{\rm box} d^4x\partial_\mu\left[i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\phi-i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\phi^*\right]\,. \tag{23.10}$$

- Since the shape and size of the box are arbitrary, the integrand must vanish:

$$\partial_\mu\left[i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\phi-i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\phi^*\right]=0 \tag{23.11}$$

Symmetry and Conserved Quantities

- Let's denote the term in brackets in (23.11) as

$$j^\mu(x) = i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\phi - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}\phi^* \tag{23.12}$$

- Using the explicit action (23.1), j^μ becomes

$$j^\mu = i\phi^*\partial^\mu\phi - \mathrm{c.c.} = 2\mathrm{Im}\phi\partial^\mu\phi^*\,, \tag{23.13}$$

  where $\mathrm{c.c.}$ denotes the complex conjugate
- With this notation, (23.11) becomes the classical conservation law for the current density j^μ:

$$\partial_\mu j^\mu = 0 \tag{23.14}$$

Symmetry and Conserved Quantities

- Writing $j^\mu = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix}$ we have

$$\partial_\mu j^\mu = \partial_0 \rho + \nabla \cdot \mathbf{j} = 0\,. \tag{23.15}$$

- Integrating over an infinite spatial volume $\int d^3x$ we have

$$\partial_0 \int d^3x \rho = 0\,, \tag{23.16}$$

where we have assumed that $\mathbf{j}$ falls sufficiently fast at infinity

Symmetry and Conserved Quantities

- We found that the symmetry (23.2) gives rise to a *conserved charge*

$$Q \equiv \int d^3x \rho \tag{23.17}$$

- This conservation law is *classical*
- The associated current density j^μ is called the *Noether current*
- The *Noether theorem* stating that **every continuous symmetry corresponds to a conserved quantity** is an extremely important result in physics
- While we only discussed the particular case (23.2), another important case is the invariance of S under translations, giving rise to the **conservation of energy** and **conservation of momentum**