Symmetry and Conserved Quantities II

paul.romatschke@colorado.edu

Fall 2020

Review

- In lecture 22, we found that the action S for a complex scalar field ϕ was invariant under the transformation

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)\,, \tag{24.1}$$

  with arbitrary (but constant) α
- In lecture 23, we discussed the Noether theorem, stating that continuous symmetries of the classical action give rise to conserved quantities
- For the complex scalar field, we found the *classical* conservation law

$$\partial_\mu j^\mu = 0\,, \tag{24.2}$$

  for the Noether current $j^\mu = 2\mathrm{Im}\left[\phi^* \partial^\mu \phi\right]$
- What about the *quantum version* of this conservation law?

Quantum Field Theory in Minkowski Space

- We have set up quantum field theory through the partition function

$$Z = \int \mathcal{D}\phi e^{-S_E}\,, \tag{24.3}$$

where S_E is the Euclidean action of the classical field theory

- This *Euclidean* field theory is well defined because integrals are convergent using imaginary time
- We can *formally* obtain results for Minkowski space by *analytically continuing* to real time as in lecture 16
- As in Eq. (5.13), we *formally* get

$$Z = \int \mathcal{D}\phi e^{iS}\,, \tag{24.4}$$

where S is the Minkowski action, e.g. (23.1)

Quantum Conservation Laws

- For the complex scalar field, the partition function is given by

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*]} \tag{24.5}$$

- The classical Minkowski action S is invariant under the symmetry (24.1)
- Let us now investigate what happens if the symmetry (24.1) also applies to the full quantum field theory
- If (24.1) is a symmetry of QFT, the partition function is invariant under the symmetry, and we can write

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{iS[\phi,\phi^*]} = \int \mathcal{D}\phi' \mathcal{D}\phi'^* e^{iS[\phi',\phi'^*]} \tag{24.6}$$

- For the symmetry (24.1), the action is invariant under the transformation $\phi \to \phi'$:

$$\delta S = S[\phi', \phi'^*] - S[\phi, \phi^*] = 0\,. \tag{24.7}$$

- In (23.7) we found that using the equations of motion, for *any* small change of ϕ

$$\delta S = \int_x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}\delta\phi^*\right]\,. \tag{24.8}$$

- Let's consider generalizing the symmetry (24.1) to

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,, \tag{24.9}$$

with $\alpha(x)$ now *position-dependent*

Quantum Conservation Laws

- For $\alpha(x)$ small, we therefore have

$$\delta\phi = i\alpha(x)\phi(x) \tag{24.10}$$

- Using $\delta\phi = i\alpha(x)\phi(x)$, (24.8) implies

$$\delta S = \int_x \partial_\mu \left(\alpha(x)\left[i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\phi - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\phi^*\right]\right) \tag{24.11}$$

- We recognize the object in brackets as the classical current density j^μ such that

$$\delta S = \int_x \partial_\mu\left(\alpha(x)j^\mu(x)\right) = \int_x \alpha\partial_\mu j^\mu + \int_x j^\mu\partial_\mu\alpha \tag{24.12}$$

Quantum Conservation Laws

- Using the fact that the classical current is conserved $\partial_\mu j^\mu = 0$, we have

$$\delta S = \int_x j^\mu(x)\partial_\mu \alpha(x) \tag{24.13}$$

- As a consequence, we have

$$e^{iS[\phi',\phi'^*]} = e^{iS[\phi,\phi^*]+i\delta S} = e^{iS[\phi,\phi^*]}\left(1 + i\int_x j^\mu \partial_\mu \alpha\right) \tag{24.14}$$

- Moreover, from (24.9) we have

$$\mathcal{D}\phi' = \mathcal{D}\phi e^{i\alpha(x)}\,, \quad \mathcal{D}\phi'^* = \mathcal{D}\phi^* e^{-i\alpha(x)}\,, \tag{24.15}$$

such that the measure $\mathcal{D}\phi'\mathcal{D}\phi'^*$ is invariant under (24.9)

Quantum Conservation Laws

- Recall that δS vanishes for any small $\delta\phi$ because it is an integral over a total derivative, cf. (24.11)
- As a consequence, the action is invariant for the generalized symmetry $\phi \to e^{i\alpha(x)}\phi$ *to linear order* in the parameter α
- Therefore, $\int \mathcal{D}\phi\mathcal{D}\phi^* e^{iS} = \int \mathcal{D}\phi'\mathcal{D}\phi'^* e^{iS'}$ holds to linear order in α, and hence

$$0 = \int_x \partial_\mu \alpha(x) \int \mathcal{D}\phi\mathcal{D}\phi^* e^{iS} j^\mu(x) = \int_x \partial_\mu \alpha(x) \langle j^\mu(x)\rangle_{\rm full} \quad (24.16)$$

- Integration by parts yields

$$0 = \int_x \alpha(x) \partial_\mu \langle j^\mu(x)\rangle_{\rm full} \quad (24.17)$$

Quantum Conservation Laws

- While $\alpha(x)$ is small, it can still be arbitrary
- For instance, we could expand $\alpha(x)$ in a complete basis set of functions with arbitrary coefficients
- Since the integral over α has to vanish for any such choice, the only possibility is that the integrand itself is vanishing:

$$0 = \partial_\mu \langle j^\mu(x) \rangle_{\rm full} \tag{24.18}$$

Quantum Conservation Laws

- We find that in quantum field theory, *the expectation value* of the current operator is conserved,

$$\partial_\mu \langle j^\mu \rangle_{\rm full} = 0\,. \tag{24.19}$$

- This generalizes the classical conservation law (24.2)