Symmetry and Conserved Quantities III

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Review

- In lecture 24, we discussed how symmetries give rise to conserved quantities in QFT
- In particular, we discussed the current conservation

$$\partial_\mu \langle j^\mu \rangle_{\rm full} = 0\,. \tag{25.1}$$

  arising from the symmetry $\phi \to e^{i\alpha}\phi$ for a complex scalar field
- In this lecture, we continue the QFT discussion and derive so-called Ward identities

Ward identities

- Besides the current conservation (25.1), we can generate additional QFT constraints on correlation functions
- These go under the name of *Ward identities* or *Ward-Takahashi identities*
- Let us derive an example of these Ward identities

Ward identities

- Instead of the partition function, consider the expectation value of the two-point function

$$\langle\phi(x)\phi^*(y)\rangle_{\rm full} = \frac{\int \mathcal{D}\phi\mathcal{D}\phi^* e^{iS}\phi(x)\phi^*(y)}{Z} \qquad (25.2)$$

- Just changing the label of the fields $\phi \to \phi'$ without actually transforming them we have

$$\frac{\int \mathcal{D}\phi\mathcal{D}\phi^* e^{iS}\phi(x)\phi^*(y)}{Z} = \frac{\int \mathcal{D}\phi'\mathcal{D}\phi'^* e^{iS'}\phi'(x)\phi'^*(y)}{Z} \qquad (25.3)$$

Ward identities

- Using the *local* transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,, \tag{25.4}$$

we have for small α

$$\phi'(x)\phi'^*(y) = \phi(x)\phi^*(y)(1 + i\alpha(x) - i\alpha(y)) \tag{25.5}$$

- Proceeding as in (24.14), we have

$$0 = \langle\phi(x)\phi^*(y)\left[\left(1 + i\int d^4z j^\mu \partial_\mu \alpha(z)\right)(1 + i\alpha(x) - i\alpha(y)) - 1\right]\rangle_{\rm full} \tag{25.6}$$

Ward identities

- Expanding in small α leads to

$$0 = \int_z \langle \phi(x)\phi^*(y) j^\mu(z)\partial_\mu \alpha(z)\rangle_{\rm full} + \langle \phi(x)\phi(y)\alpha(x)\rangle_{\rm full} - \langle \phi(x)\phi(y)\alpha(y)\rangle_{\rm full}\,. \tag{25.7}$$

- Writing $\alpha(x) = \int_z \delta(z-x)\alpha(z)$ and performing integration by parts on the first term, we have

$$0 = \int_z \alpha(z)\left[-\partial_\mu \langle \phi(x)\phi^*(y) j^\mu(z)\rangle_{\rm full} + \langle \phi(x)\phi^*(y)\rangle_{\rm full}\delta(x-z) - \langle \phi(x)\phi^*(y)\rangle_{\rm full}\delta(y-z)\right]\,. \tag{25.8}$$

- Since $\alpha(z)$ is again arbitrary, the integrand must vanish

Ward identities

- As a consequence, we find a *Ward-identity* for the complex scalar field:

$$
\begin{aligned}
\partial_\mu \langle \phi(x)\phi^*(y) j^\mu(z)\rangle_{\rm full} &= \langle \phi(x)\phi^*(y)\rangle_{\rm full}\delta(x-z)\\
&\quad -\langle \phi(x)\phi^*(y)\rangle_{\rm full}\delta(y-z)\,, \quad (25.9)
\end{aligned}
$$

  where the derivative on the lhs is acting on the coordinate z only.
- The Ward identity is best represented in momentum space

Ward identities

- Because of translational invariance we have

$$\langle\phi(x)\phi^*(y)\rangle_{\rm full} = \langle\phi(x-y)\phi^*(0)\rangle_{\rm full} = G_{\rm full}(x-y) \tag{25.10}$$

  where $G_{\rm full}$ is the full propagator of the theory

- Fourier transforming $G_{\rm full}(x-y)\delta(x-z)$ we have

$$\begin{aligned}\int_{x,y} e^{-i(P+K)\cdot x - iQ\cdot y} G_{\rm full}(x-y) &= \int_{x,y} e^{-i(P+K)\cdot x - i(P+K+Q)\cdot y} G_{\rm full}(x)\,,\\ &= \int_y e^{-i(P+K+Q)\cdot y}\tilde{G}_{\rm full}(P+K)\,,\\ &= \delta(P+K+Q)\tilde{G}_{\rm full}(P+K) \quad (25.11)\end{aligned}$$

Ward identities

- Fourier-transforming the rhs of the Ward-identity (25.9) gives the full propagator and an overall δ function corresponding to momentum conservation
- Momentum conservation will also give rise to the same δ-function on the lhs of (25.9)
- Skipping the δ function, the Fourier-transform of (25.9) gives

$$lhs = \tilde{G}_{\rm full}(P+K) - \tilde{G}_{\rm full}(P) \tag{25.12}$$

- A little more work is needed to recognize the lhs to be related to the full 3-vertex $\Gamma_{3,{\rm full}}$ of the theory

Ward identities

- Ward identities give relations between *full* correlation functions of the theory
- For the complex scalar field, (25.9) is a relation between the full propagator and the full 3-vertex
- These relations do *not* rely on a perturbative expansion
- Ward identities are *exact* relations in QFT